Asymmetric Auctions with More Than Two Bidders

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Abstract

The theoretical literature on asymmetric first-price auctions has focused mainly on settings with either (1) exactly two bidders or (2) an arbitrary number of bidders with types in a common support. Even though closed form solutions are typically impossible, there is enough structure in the problem to guide both empirical work and numerical solutions or simulations. However, casual observation of real-world auctions suggests that the assumptions are restrictive. Indeed, the empirical literature has ventured beyond these models. We explain the relevant complications that arise if the above conditions do not hold, emphasizing that the structure of the problem changes significantly. Critically, the dimensionality of the problem appears to increase as bidders may now tender bids over different supports (bid bifurcation). The main conceptual contribution of this paper is to establish a method by which the dimensionality of the problem can once again be reduced. This insight is used to construct the first robust solution algorithm that allows for bid bifurcation. Accurate solution methods are essential to e.g. evaluate different counterfactual policies. We provide sufficient conditions, and in some cases necessary and sufficient conditions, for bid bifurcation to occur and demonstrate its relevance through a series of analytical and numerical examples. Implications for empirical work are emphasized.

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1 Introduction

Although auction theory is often touted as a success story, there remains important areas where the theory is less than fully developed. One such area is asymmetric auctions—auctions involving bidders who are in some way different from each other. Such settings are typically modeled as involving heterogeneous bidders who draw types from different distributions. This is not the only case which would result in an asymmetric auction as they could also arise because of collusive efforts from a subset of bidders, if bidders have different preferences (for example, different risk attitudes), if they face different constraints (financial or, perhaps in a procurement setting, capacity-based), or even because the awarding rule treats the bids from different groups asymmetrically (as in the case of bid preference policies). Aided by the richness of available auction data, empirical researchers have noted potential sources of asymmetry, particularly in the procurement setting, based off experience in the industry, region in which the bidders are located, distance to project locations, backlog, or classification of the bidder (firm) such as whether it is a small business, owned by a minority, veteran, or female, for example.[1] The literature on the structural econometrics of auctions has married this detailed data with theoretic models and developed rapidly since Paarsch's (1992) seminal work, increasing the demand for theoretical results which characterize equilibrium; see Hickman, Hubbard, and Sağlam (2012) for a recent survey. Even when bidder identities are not observed (or are only partially observed), Lamy (2012) provides a way to still allow for asymmetries in estimation.

Unfortunately, some important issues have gone overlooked as empirical researchers have tried to account for asymmetries but, in some cases, ventured into territory not yet fully explored by theory. The purpose of our work is to point out a potential oversight in such analysis and, more constructively, to further develop the theory of asymmetric auctions to help empirical researchers tackle the particular stumbling block identified here. To understand the issue, it is helpful to first identify a weakness in the existing theoretical literature on asymmetric first-price auctions.[2] There are

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[1]To give a few examples, Campo, Perrigne, and Vuong (2003) distinguish between joint and solo bidders, De Silva, Dunne, and Kosmopoulou (2003) classify bidders as entrant or incumbents, Flambard and Perrigne (2006) use firm location as a source of asymmetry, Marion (2007) as well as Krasnokutskaya and Seim (2011) both identify state-qualified small business enterprises and non-small businesses, Athey, Levin, and Seira (2011) consider loggers and sawmills as inherently different, and De Silva, Hubbard, and Kosmopoulou (2015) see firm eligibility status and election of whether to complete a state-run training program.

[2]Most of the theoretical literature is in the context of standard auctions in which buyers pay to

two common models. In the first model there are exactly two bidders, where typically a "weak" bidder is facing a "strong" bidder. It is not hard to see that in equilibrium the two bidders share a common maximum bid; otherwise, one bidder could lower his bid without lowering his winning probability. In comparison, empirical researchers might study auctions with a number of weak bidders and a number of strong bidders; as examples, the weak bidders might be characterized as small, inexperienced, having a high capacity utilized, unfavorably located, or otherwise depending on the application, with strong bidders taking on the opposite characteristic(s). In the second model, there is an arbitrary number of bidders, but the distributions of bidders' types have a common support. In this case, it can also be shown that all bidders share a common maximum bid. Examples of the first kind of model are Maskin and Riley (2000) and Kirkegaard (2012), while examples of the second kind are Lebrun (1999) and Kirkegaard (2009), among many others. Technically, these models have the helpful feature that there is a single boundary condition. This makes it easier for theorists to make inferences from the system of differential equations describing equilibrium behavior.

To highlight the limitations of such models, imagine two weak bidders are facing two strong bidders. For instance, two art students are up against two well-known billionaire art collectors in a first-price auction for an old masterpiece. The latter are unlikely to be too concerned about the presence of the former, and it is patently absurd to suggest that the two kinds of bidders would share the same maximum bid. Compared to the first kind of model mentioned above, the issue here is that once the economy grows, equilibrium behavior undergoes a qualitative change. Stated differently, replicating the economy is not without its pitfalls. Compared to the second kind of model noted above, the issue is that the type supports are likely very different for the students and billionaires. In either situation, it is no longer necessarily the case that there is a single boundary condition prescribing that all bidders tender the same highest bid. In these situations, any analysis that assumes a single boundary condition should thus be treated with some caution.

It is pertinent to note that empirical researchers typically assume bidders' types are drawn from distributions that share a common, compact support having strictly

acquire an item. To ease comparison with this literature, we use the same setting and terminology here. However, much of the empirical work is based off procurement settings in which firms represent bidders vying for the right to complete a task for the seller (typically a government) at the lowest cost. This difference is discussed further below.

positive densities; for example, most structural work employs some version of the two-step estimator proposed by Guerre, Perrigne, and Vuong (2000) which maintains these assumptions. Such an estimator is often extended and applied in asymmetric settings. For example, Flambard and Perrigne (2006) consider snow removal contracts in Montreal. In Figure 1 of their paper, they depict distributions of bids submitted (per meter) in certain tracts based on where the firm tendering the offer is located. Strong bidders are quite frequently observed submitting bids that are far more aggressive (i.e., lower, in the case of procurement) than even the most aggressive bid ever submitted by weak bidders which suggests that bidding supports do not overlap. Still, to our knowledge, all empirical researchers have maintained the assumption of a common type support—perhaps because theory is most well understood in that context.

This complication has been recognized in the theoretical literature before. However, the only paper that tackles the problem head on appears to be Lebrun (2006), but the point of that paper is solely to establish uniqueness of the equilibrium. It is harder to find explicit reference to the problem in the empirical literature, but Athey and Haile (2007, pages 3885–3886) provide a brief discussion that suggests (1) researchers in the existing literature maintain the assumption of common type support, (2) conditions for observed bids to be rationalized by equilibrium behavior (such as the common high bid) are critical, and (3) "plausible specifications of primitives" (such as the bounds of the type supports) might violate the common high bid assumption, "so it may be useful to relax that assumption in practice." Despite the warning and discussion they provide, to our knowledge no one has pursued this venture and we hope to allow for this. Lastly, because analytic solutions to the standard system of differential equations describing equilibrium behavior rarely exist, researchers often solve for the bidding strategies numerically. However, researchers contributing to that literature have also not dealt with the problem of potentially different bid supports. In fact, the opposite is often true—numerical approaches often hinge critically on the common bid support. Hubbard and Paarsch (2014, footnote 8) recognize that the canonical boundary conditions may need adjustment if the type supports differ when there are more than two bidders at auction; however, they do not consider such an extension.

Thus, this particular analytical problem has been recognized in some corners of the literature. We maintain however, that the problem is not widely known among

practitioners and that no one has considered how to accommodate these empirically-relevant settings. Note, too, that extending models to allow for different bid supports is important from a policy perspective. As an example, take the case of bid preference policies which has received much attention due to the prevalence of such programs in procurement auctions; see, for example, Marion (2007), Hubbard and Paarsch (2009), and Krasnokutskaya and Seim (2011). We outline such a policy in a first-price setting and note that the asymmetry is effectively endogenously determined by the size of the preferential treatment. Specifically, in a standard first-price auction, a bidder's expected utility if his valuation is v and his bid is b is $(v - b)G(b)$, where $G(b)$ is the (endogenously determined) winning probability. Now assume this bidder is given preferential treatment. For instance, if he wins he has to pay only a fraction $r \in (0, 1)$ of his bid. As bidding strategies are likely to change, the winning probability for any given bid is likely to change as well. Thus, let $G_r(b)$ denote the new winning probability, given b and r. The bidder's expected utility is now $(v - rb)G_r(b)$, which is of course maximized where $(v/r - b)G_r(b)$ is maximized. Thus, from his competitors' point-of-view, giving the bidder preference is equivalent to changing the distribution of his types. In particular, the support shifts to the right. Even if all bidders start out with the same support, preferential treatment effectively destroys that property. If the bid preference is sufficiently large, or if one would like to simulate behavior under such policies (such as in counterfactual experiments), one may be concerned that equilibrium behavior is no longer accurately characterized by a single boundary condition.

In this paper, we investigate an asymmetric auction model which allows for more than two bidders. The critical feature of these settings is whether equilibrium bids are projections from types into a common support, or not. When there exists a region in which the bidding supports do not overlap, we say that bid bifurcation occurs.[3] The possibility of bid bifurcation presents new challenges. At the conceptual level, bid bifurcation seems to increase the dimensionality of the problem as different bidders may now have different maximum bids. The main contribution of the first part of the paper is to advance the theory to the point where the dimensionality of the problem can be reduced. Focusing on the case with two distinct groups of bidders, we show

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[3]We borrow and adapt the term from Parreiras and Rubinchik (2010), who consider related complications in all-pay auctions with many bidders. See the conclusion for further discussion of all-pay auctions as well as the possibility of more groups of bidders.

that the maximum bids of the two groups are mechanically linked in a way that implies that it is sufficient to only worry about the lowest of the maximum bids. Conceptually, the problem is then no harder than the standard problem with a single boundary condition.

It is obviously legitimate to ask whether bid bifurcation is merely a theoretical curiosity or whether it might play a role in real-world auctions and by extension in empirical work. We present comparative statics results which prove that bid bifurcation is more likely to occur the "stronger" bidders are (in terms of the reverse hazard rate). When distributions are uniform over different supports, we verify the intuition that bid bifurcation is more likely the more bidders there are. Indeed, in this case it is possible to quantify precisely how large the asymmetry between bidders must be in order for bid bifurcation to arise. Stated differently, we derive necessary and sufficient conditions for bid bifurcation in this environment. For example, if there are three "strong" bidders and two "weak" bidders, with supports $[0, \overline{v}_1]$ and $[0, \overline{v}_2]$ respectively, bid bifurcation results as long as $\overline{v}_1$ is more than 8.11% percent larger than $\overline{v}_2$. This percentage rapidly decreases in the number of bidders.[4] Our interpretation is that these asymmetries are so small that one should not a priori dismiss the possibility of bid bifurcation in real-world applications. Sufficient conditions for bid bifurcation in the general model (without uniform distributions) are also presented.

Having now argued that bid bifurcation may be a real concern, it is important to realize that there are ramifications for both empirical work and for the algorithms typically used to obtain numerical solutions. Note that the solution approach often links theory and estimation. For example, a parametric structural approach in the spirit of Donald and Paarsch (1993,1996) applied to an asymmetric auction would require solving for the equilibrium for every guess of the parameters in a pseudo maximum likelihood routine. Moreover, regardless of the estimation strategy, such solutions are an essential ingredient in counterfactual policy analysis or simulations involving asymmetric first-price auctions. Thus, it is desirable to build a robust algorithm that takes bid bifurcation into account. The second part of the paper is devoted to this endeavour. Here, the reduction in the dimensionality of the problem that we obtain in the first part of the paper once again plays a crucial role.

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[4]The model in which bidders' types are uniformly distributed over different supports has been extremely valuable to theorists since Vickrey's (1961) groundbreaking work. Kaplan and Zamir (2012) recently generalized Vickrey's equilibrium characterization. Both papers assume there are precisely two bidders. We continue in this tradition, but allow for more bidders.

Because we know much about the uniform distribution setting, it provides an ideal testbed for our general numerical solution strategy. Thus, we first use uniform examples to substantiate that the algorithm performs well before providing other solved examples which build off our earlier results. We provide a discussion of our results and note how they can be useful to applied researchers in Section 5.

2 Asymmetric auctions with many bidders

A total of n risk neutral bidders are participating in an independent private-values first-price auction. Bidder i's value or type, v_i, is drawn from the distribution function F_i, with continuous and strictly positive density f_i and support $[\underline{v}_i, \overline{v}_i]$, $\overline{v}_i > \underline{v}_i \geq 0$. Lebrun (2006) has proven that there is an essentially unique equilibrium in undominated strategies under the mild assumption that F_i is strictly log-concave near $\underline{v}_i$, $i = 1, 2, ..., n$.[5] In this paper we simply assume that the equilibrium is unique. Bids are continuous and strictly increasing in type among those types that have a strictly positive probability of winning. Let $\overline{b}_i$ denote the bid submitted by bidder i with type $\overline{v}_i$. We begin with an intuitive observation.

Lemma 1 *If* $\overline{v}_i = \overline{v}_j$ *then* $\overline{b}_i = \overline{b}_j$. *If* $\overline{v}_i > \overline{v}_j$ *then* $\overline{b}_i \geq \overline{b}_j$.

Proof. See the Appendix. ■

The focus of the paper is the possibility that $\overline{b}_i > \overline{b}_j$. The current section describes some first theoretical insights. These are then utilized to develop a solution procedure that can correctly handle the case where $\overline{b}_i > \overline{b}_j$. As we discuss in Section 4, this procedure extends current approaches which can, and should, be amended given our results.

Let $\varphi_i(b)$ denote bidder i's inverse bidding strategy. Assume, for now, that there is a range of bids where all bidders are active. If bidder i with type v contemplates bidding in this range, his expected payoff is $(v - b) \prod_{j \neq i} F_j(\varphi_j(b))$, which is maximized where

$$\ln (v - b) + \sum_{j \neq i} \ln F_j(\varphi_j(b))$$

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[5]That is, the reverse hazard rate $\frac{f_i(v)}{F_i(v)}$ is strictly decreasing on an interval $(\underline{v}_i, \underline{v}_i + \delta)$, $\delta > 0$. The assumption can be further weakened if $\underline{v}_i$ is not the same for all i.

is maximized. Deriving the first order condition and imposing the equilibrium condition that $v = \varphi_i(b)$ produces

$$\sum_{j \neq i} \frac{d}{db} \ln F_j(\varphi_j(b)) = \frac{1}{\varphi_i(b) - b}. \tag{1}$$

Summing (1) across all agents and subtracting (1) for agent i yields the system of differential equations

$$\frac{d}{db} \ln F_i(\varphi_i(b)) = \frac{1}{n-1} \left[\sum_{j \neq i} \frac{1}{\varphi_j(b) - b} - \frac{n-2}{\varphi_i(b) - b} \right]. \tag{2}$$

It follows from Lemma 1 that if $\overline{v}_i = \overline{v}$ for all i, then there is some $\overline{b}$ such that $\varphi_i(\overline{b}) = \overline{v}_i$, $i = 1, n$. Hubbard and Paarsch (2014) discuss this system at length with a focus on detailing and comparing ways in which researchers have gone about solving this system which rarely admits a closed-form solution. As we have noted, the approaches considered are generally valid only if $n = 2$ (even if $\overline{v}_1 \neq \overline{v}_2$) or if $\overline{v}_i = \overline{v}$ for all bidders $i = 1, 2, ..., n$.

However, the point of the current paper is to allow $n > 2$ and $\overline{v}_i \neq \overline{v}_j$ for some (i, j) pair. For concreteness, and in line with the relevant empirical literature, assume in the remainder that bidders draw types from one of two distribution functions.[6] Bidders $1, ..., m_1$ draw types from the distribution F_1, while bidders $m_1 + 1, ..., n$ all draw types from F_n. Let $m_n = n - m_1$ denote the number of bidders in the latter group. It follows from Lebrun (2006) that bidders within each of the two groups use symmetric strategies. Thus, the first m_1 bidders all use the same inverse bidding strategy as bidder 1, $\varphi_1(b)$. Likewise, the last m_n bidders are all ex ante identical to bidder n, and thus they all use the strategy $\varphi_n(b)$.

The complete solution to an asymmetric auction model requires appropriate boundary conditions. As such, we continue with an overview of these conditions, for a few reasons. First, the wrong boundary condition can lead to either the wrong solution or no solution at all. Second, boundary conditions often drive the approach employed in solving for the equilibrium strategies. Lastly, for convenience and completeness, we collect in one place all the different scenarios that one may encounter.

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[6]The only empirical paper we know of that treats more than two different groups of bidders is De Silva, Hubbard, and Kosmopoulou (2015). However, due to data constraints, they only deal with auctions in which combinations of two groups of bidders are present in their application.

By Lemma 1, $\bar{b}_1 = \bar{b}_n$ if $\bar{v}_1 = \bar{v}_n$. Of course, the case where $\bar{v}_1 \neq \bar{v}_n$ is the most interesting for our purposes. Thus, assume $\bar{v}_1 > \bar{v}_n$. Then, $\bar{b}_1 \geq \bar{b}_n$. Nonetheless, all bidders must share the same maximum bid if $m_1 = 1$. The reason is familiar; if $\bar{b}_1 > \bar{b}_n$, then bidder 1 with type $\bar{v}_1$ could lower his bid slightly and still win with probability one. Hence, assume from now on that $m_1 \geq 2$. The boundary conditions are that $\varphi_i(\bar{b}_i) = \bar{v}_i$, $i = 1, n$.

Next, assume that the supports strictly overlap, such that $[\underline{v}_1, \bar{v}_1] \cap [\underline{v}_n, \bar{v}_n]$ has strictly positive measure. With no overlap, or $\underline{v}_1 \geq \bar{v}_n$, the existence of the last m_n bidders is irrelevant as they would have no chance of winning the auction (at a price that does not exceed their valuation). In other words, the problem is interesting only if the supports overlap. We have $\bar{v}_1 > \bar{v}_n > \underline{v}_1, \underline{v}_n$.

At this juncture, an examination of bidding behavior among low types is required as this will provide the initial conditions. Imagine first that $\underline{v}_1 > \underline{v}_n$. Since $m_1 \geq 2$, competition among the first group of bidders ensure that they will bid at least $\underline{v}_1$. Thus, bidder n wins with probability zero if his type is below $\underline{v}_1$. Similarly, if $m_n \geq 2$ and $\underline{v}_n > \underline{v}_1$ then bidders in the first group stand no chance of winning if their types are below $\underline{v}_n$. Stated differently, in both cases any bidder has a strictly positive chance of winning if and only if his type strictly exceeds $\underline{v} = \max\{\underline{v}_1, \underline{v}_n\}$. This insight provides the "initial condition" that $\varphi_i(\underline{v}) = \underline{v}$, $i = 1, n$.

However, the above discussion does not include the possibility that $m_n = 1$ and $\underline{v}_n > \underline{v}_1$. In this case, it is no longer true that bidder n of type $\underline{v} = \underline{v}_n$ bids his true value. There is an incentive to bid lower, as he may still win in the event that his rivals all have types below $\underline{v}$. Following Maskin and Riley (2000) and Lebrun (2006), the initial condition can nevertheless be uniquely and explicitly characterized. Specifically, among the first group of bidders there is a threshold value, $\underline{b} \in (\underline{v}_1, \underline{v}_n)$, such that any bidder bids his true value (and wins with probability zero) if his type is below $\underline{b}$. In equilibrium, $\varphi_n(\underline{v}_n) = \underline{b}$. Hence, bids above $\underline{b}$ are what Lebrun (2006) terms "serious bids" as they entail a strictly positive winning probability. The equilibrium value of $\underline{b}$ is

$$\underline{b} = \max\left(\arg\max_b (\underline{v}_n - b) F_1(b)^{m_1}\right). \tag{3}$$

To understand (3), recall that the first m_1 bidders bid their true value when their type is below $\underline{b}$. The expression in the parenthesis says that bidder n with type $\underline{v}_n$ must best respond to this bidding behavior.

In summary, if $m_1, m_n \geq 2$, then the initial condition is that $\varphi_i(\underline{v}) = \underline{v}$, $i = 1, n$. Here, the smallest serious bid is $\underline{b} = \underline{v}$. The same is true if $m_n = 1$ and $\underline{v}_1 > \underline{v}_n$. If $m_n = 1$ and $\underline{v}_n > \underline{v}_1$, however, the initial condition is that $\varphi_n(\underline{v}_n) = \underline{b}$ and $\varphi_1(\underline{b}) = \underline{b}$, where $\underline{b}$ is determined by (3). The initial condition is determined in a similar manner if $m_1 = 1$ and $\underline{v}_1 > \underline{v}_n$.

Figure 1 depicts some complications that can arise in the inverse bidding strategies in light of Lemma 1 and our discussion of the boundary conditions. Though we have not discussed the details of everything being presented in these graphs (we will refer back to these later in the paper), it may help to visualize some of the discussion we've presented until this point. Specifically, in panel (a) we present a situation in which (i) either $\underline{v}_1 = \underline{v}_n$ or $m_n > 1$, (ii) $\overline{v}_1 > \overline{v}_n$, and ($iii$) $\overline{b}_1 > \overline{b}_n$. Likewise, in panel (b) we depict a situation in which (i) $m_n = 1$, (ii) $\overline{v}_1 > \overline{v}_n > \underline{v}_n > \underline{v}_1$, and ($iii$) $\overline{b}_1 > \overline{b}_n$. Note that the inverse bidding strategies must cross in such a situation like that presented in panel (b). As denoted in Figure 1, let $\widehat{v} = \varphi_1(\overline{b}_n) \in (\underline{v}_1, \overline{v}_1]$ be the (endogenously determined) type of bidder 1 that submits a bid of exactly $\overline{b}_n$. Going forward, it is important to realize that $\overline{b}_1 > \overline{b}_n$ if and only if $\widehat{v} < \overline{v}_1$. That said, the relationship between $\overline{b}_1$ and $\overline{v}_n$ is not clear—panel (a) provides an example where $\overline{b}_1 > \overline{v}_n$ while panel (b) shows a scenario in which $\overline{b}_1 < \overline{v}_n$, though this relationship is not tied to the boundary condition at the lower end of the bid support.

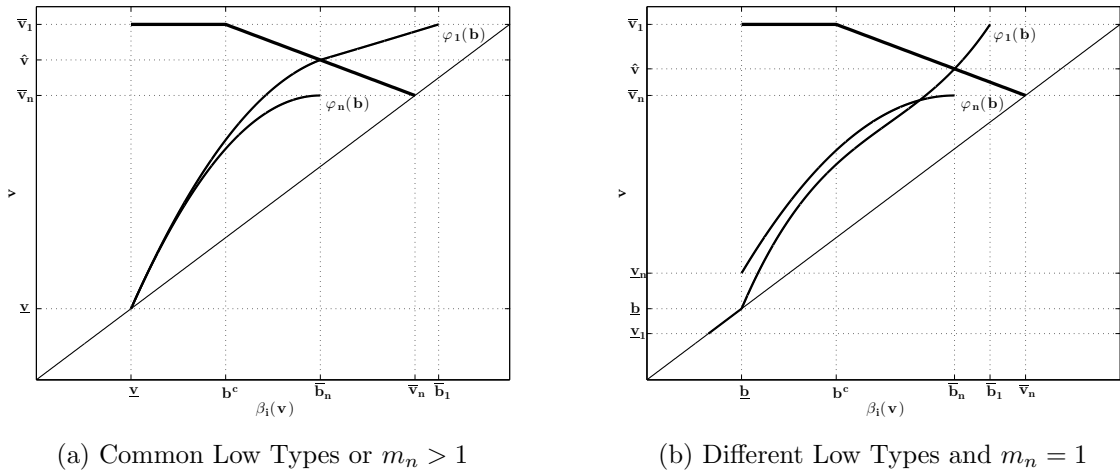(a) Common Low Types or $m_n > 1$ (b) Different Low Types and $m_n = 1$

Figure 1: Potential Inverse Bid Functions

With the boundary and initial conditions in place, we return to "interior" bids. Since strategies are continuous and strictly increasing, it follows that for each bid in $[\underline{b}, \bar{b}_i]$, bidder i has a type that submit such a bid. Thus, $[\underline{b}, \bar{b}_1] = [\underline{b}, \bar{b}_n] \cup [\bar{b}_n, \bar{b}_1]$ consists of one interval of bids where all bidders are potentially active as well as a (possibly empty) set of higher bids where only bidders in the first group compete. The system in (2) still accurately describes behavior for bids in $(\underline{b}, \bar{b}_n)$. If $\bar{b}_n < \bar{b}_1$, then (2) should be replaced by

$$\frac{d}{db} \ln F_1(\varphi_1(b)) = \frac{1}{m_1 - 1} \frac{1}{\varphi_1(b) - b} \tag{4}$$

for $b \in (\bar{b}_n, \bar{b}_1]$, since only the first m_1 bidders are active on this range. The simple form of (4) reflects the fact that at high bids the auction is essentially a symmetric auction (involving only a symmetric subset of the population).

As mentioned previously, in cases where $\bar{b}_1 = \bar{b}_n = \bar{b}$ (e.g., if either $\bar{v}_1 = \bar{v}_n$ or $m_1 = m_n = 1$) researchers typically solve for a pair $(\varphi_1(b), \varphi_n(b))$ of inverse bidding strategies along with the common high bid $\bar{b}$ (which is unknown a priori) satisfying the system (2) as well as the initial conditions. Generalizing this to the case in which bid supports may potentially differ initially seems much harder since the system (2) remains, but now we need to solve for not just a single unknown value $(\bar{b})$, but three $(\hat{v}, \bar{b}_n, \bar{b}_1)$. However, it is easy to reduce this to two free variables. For instance, given any $(\bar{b}_n, \bar{b}_1)$ pair, $\hat{v}$ can be obtained by integrating (4) backwards (which can be done analytically) from $\bar{b}_1$ to infer the v_1 value $(\hat{v})$ for which $v_1 = \varphi_1(\bar{b}_n)$. Alternatively, for any $(\hat{v}, \bar{b}_n)$ pair, $\bar{b}_1$ can be computed simply by integrating (4) forwards from $b = \bar{b}_n$, $\varphi_1(\bar{b}_n) = \hat{v}$. Although it may seem natural to consider finding a $(\bar{b}_n, \bar{b}_1)$ pair, we find it more fruitful to think of $(\hat{v}, \bar{b}_n)$ as the pair to be determined.

Still, while the system (2) is appropriate for the bid range $[\underline{b}, \bar{b}_n]$, an apparent complication is that a pair $(\hat{v}, \bar{b}_n)$ is involved, suggesting that the search for a valid boundary condition is a two-dimensional problem. However, as we will soon show, it turns out there is a one-to-one mapping between $\bar{b}_n$ and $\hat{v}$ which means that the unknown pair can in fact be reduced to a one-dimensional problem. In other words, there is in reality only one free variable in the triplet $(\hat{v}, \bar{b}_n, \bar{b}_1)$. It is worth emphasizing the point that, because of this relationship, conceptually and computationally the problem is now no more complicated than in the models where $\bar{b}_1 = \bar{b}_n$ is known to hold at the outset. In both cases, the system (2) applies and one value is unknown

beforehand—in the standard case this is $\bar{b}$, while in the general case it is $\bar{b}_n$ (which implies a value for $\widehat{v}$ and in turn a value for $\bar{b}_1$). The chief difference is that in the standard models $\widehat{v}$ is known to equal $\bar{v}_1$, whereas here we have to determine $\widehat{v}$ through a step that luckily turns out to be trivial.

To see this relationship, first note that Lebrun (2006) has shown that if $b \in (\underline{v}, \bar{b}_i)$ then $\varphi_i'(b) > 0$. The key step is now to show that if $\bar{b}_n < \bar{b}_1$ then the (left-)derivative of φ_n at $\bar{b}_n$ is zero, $\varphi_n'(\bar{b}_n) = 0$. Note that if $\bar{b}_n < \bar{b}_1$ then $\widehat{v} < \bar{v}_1$.

Lemma 2 *If $\bar{b}_n < \bar{b}_1$ then $\varphi_n'(\bar{b}_n) = 0$.*

Proof. See the Appendix. ■

Since $\varphi_n(\bar{b}_n) = \bar{v}_n$ and $\varphi_1(\bar{b}_n) = \widehat{v}$, it follows from (2) that

$$\text{sign}\{\varphi_n'(\bar{b}_n)\} = \text{sign}\left\{\frac{m_1}{\widehat{v} - \bar{b}_n} - \frac{m_1 - 1}{\bar{v}_n - \bar{b}_n}\right\}. \tag{5}$$

Starting with the assumption that $\bar{b}_n < \bar{b}_1$, Lemma 2 makes it possible to solve for $\widehat{v}$,

$$\widehat{v} = \frac{m_1}{m_1 - 1}\bar{v}_n - \frac{1}{m_1 - 1}\bar{b}_n. \tag{6}$$

Note that since any equilibrium candidate must satisfy $\bar{b}_n < \bar{v}_n$, it holds that $\widehat{v} > \bar{v}_n$. Of course, the restriction that $\widehat{v}$ cannot exceed $\bar{v}_1$ must also be taken into account. The solution in (6) satisfies $\widehat{v} < \bar{v}_1$ if and only if

$$\bar{b}_n > m_1\bar{v}_n - (m_1 - 1)\bar{v}_1, \tag{7}$$

the right hand side of which is smaller than $\bar{v}_n$ whenever $\bar{v}_n < \bar{v}_1$. Indeed, from (2), if $\bar{b}_n \in (m_1\bar{v}_n - (m_1 - 1)\bar{v}_1, \bar{v}_n)$ then $\varphi_1(\bar{b}_n) = \bar{v}_1$ would imply $\varphi_n'(\bar{b}_n) < 0$, which is inconsistent with an equilibrium. Thus, for $\bar{b}_n$ candidates in this range, it *must* be the case that $\bar{b}_n < \bar{b}_1$, meaning that $\widehat{v}$ is determined by (6). Values of $\bar{b}_n$ for which $\bar{b}_n \geq \bar{v}_n$ are inconsistent with an equilibrium. This leaves the possibility that $\bar{b}_n \leq m_1\bar{v}_n - (m_1 - 1)\bar{v}_1$, at least in the case where the right hand side exceeds $\underline{v}$. However, if $\bar{b}_n < m_1\bar{v}_n - (m_1 - 1)\bar{v}_1$ then the candidate in (6) would exceed $\bar{v}_1$. Stated differently, there are no feasible values of $\varphi_1(\bar{b}_n)$ for which $\varphi_n'(\bar{b}_n)$ is not strictly positive. Then, by Lemma 2, any such candidate must thus satisfy $\bar{b}_n = \bar{b}_1$, or equivalently $\widehat{v} = \bar{v}_1$. In summary, $\widehat{v}$ has been characterized for each $\bar{b}_n$ candidate,

$\bar{b}_n \in (\underline{b}, \bar{v}_n)$. Returning to the situations presented in the panels of Figure 1, the piecewise linear function in each subplot depicts this relationship. Note that the intersection of this line with the inverse bidding strategy $\varphi_1(b)$ obtains at the point $(\bar{b}_n, \widehat{v})$ in both figures. We state this insight formally as a proposition.

Proposition 1 *For any* $\bar{b}_n \in (\underline{b}, \bar{v}_n)$, $\widehat{v}$ *is uniquely determined by*

$$\widehat{v} = \min \left\{ \bar{v}_1, \frac{m_1}{m_1 - 1} \bar{v}_n - \frac{1}{m_1 - 1} \bar{b}_n \right\}. \tag{8}$$

Proof. In text. ∎

Proposition 1 is what simplifies the dimensionality of this general problem, making the complexity of its solution akin to that of the common bid support setting as we suggested earlier. In the general case, there are evidently two possibilities: either $\widehat{v} = \bar{v}_1$ (no bid bifurcation) or $\widehat{v} < \bar{v}_1$ (bid bifurcation occurs). Given (8), the latter holds if and only if $\bar{b}_n$ exceeds the critical value b^c, where

$$b^c = m_1 \bar{v}_n - (m_1 - 1)\bar{v}_1, \tag{9}$$

which is indicated in the subplots of Figure 1 as the kink points in the piecewise functions. If $\bar{b}_n \leq b^c$, then $\widehat{v} = \bar{v}_1$ and there is no bid bifurcation. As such, there are precisely two mutually exclusive possibilities: either (*i*) $\bar{b}_1 = \bar{b}_n \leq b^c$, or (*ii*) $\bar{b}_1 > \bar{b}_n > b^c$.

In equilibrium, $\bar{b}_n > \underline{b}$. Thus, if $\underline{b} > b^c$ then $\bar{b}_n > \underline{b} > b^c$ must hold. In this case, bid bifurcation must occur in equilibrium. Stated differently, $\underline{b} > b^c$ is sufficient for bid bifurcation (but not necessary).

Corollary 1 *In equilibrium,* $\bar{b}_n < \bar{b}_1$ *if* $\underline{b} > b^c$.

Proof. In text. ∎

Corollary 1 reveals, as is intuitive, that bidding must break into two regions, or $\bar{b}_n < \bar{b}_1$, if $\bar{v}_1$ is sufficiently large compared to $\bar{v}_n$. Note also that b^c is decreasing in m_1. In other words, bid bifurcation must occur if m_1 is large enough. The intuition behind the link between the number of bidders and bid bifurcation is explained in more detail in Section 3.1.

A complementary sufficient condition for bid bifurcation can be derived. To set the scene, consider the example from the introduction in which two billionaire art

collectors compete against two art students. Let $\bar{b}_1^s$ denote the maximum bid in a (counterfactual) symmetric auction involving only the billionaires. It is a standard exercise to obtain $\bar{b}_1^s$ as it is derived from a symmetric auction. That is, $\bar{b}_1^s$ need not be known to solve the (in this symmetric case) equation characterizing equilibrium behavior. Imagine for the moment that $\bar{b}_1^s > \bar{v}_n$, where $\bar{v}_n$ denotes the highest possible type among the art students. Once the students enter the auction, it seems unlikely that the billionaires will accommodate them by lowering their maximum bid. In this case, bid bifurcation must occur, as even a student of type $\bar{v}_n$ will be unwilling to bid above $\bar{b}_1^s$. This style of argument can be further refined. Thus, it turns out that bid bifurcation must take place if $\bar{b}_1^s$ exceeds b^c. The reason is that $\bar{b}_1 > \bar{b}_1^s$; i.e., the highest bid increases when the last m_n bidders enter the auction. Thus, $\bar{b}_1 > \bar{b}_1^s > b^c$. However, as noted after (9), $\bar{b}_1 > b^c$ occurs only in the case of bid bifurcation.

Corollary 2 *In equilibrium, $\bar{b}_n < \bar{b}_1$ if $\bar{b}_1^s > b^c$.*

Proof. See the Appendix. ∎

Note that if $\underline{v}_1 > \underline{v}_n$ then $\bar{b}_1^s > \underline{v}_1 = \underline{b}$, in which case Corollary 2 is stronger than Corollary 1.

In the next section we develop further, more specific, theoretical results, focusing again on when bid bifurcation is likely to occur. The section employs uniform distributions which will later serve as a nice way of verifying the performance of solutions to asymmetric auction problems.

3 The incidence of bid bifurcation

The current section contains two main results. The first result identifies precisely when bid bifurcation occurs in the special case when both distributions are uniform distributions. This result is thus of a quantitative nature. The second result proves that bid bifurcation is more likely to occur when either group of bidders becomes "stronger" in a standard auction-theoretical sense. Thus, this can be seen as a qualitative result, or as a comparative statics exercise. Combining the two results, and thinking of the uniform distribution as a benchmark, then produce further insights into how often bid bifurcation occurs.

3.1 The uniform benchmark

Assume both distributions are uniform, on different supports. The uniform distribution satisfy Lebrun's (2006) condition, implying the equilibrium is unique. It turns out that the exact values of $\widehat{v}$, $\overline{b}_n$, and $\overline{b}_1$ can be derived analytically. Thus, it is possible to describe precisely when $\overline{b}_n < \overline{b}_1$. The proof, in the appendix, may be of some independent interest. The proof demonstrates that in the uniform case, insights from mechanism design theory can be used to obtain another, quite separate, characterization of the $(\widehat{v}, \overline{b}_n)$ pair. Combined with the characterization above, it is then possible to solve explicitly for both $\widehat{v}$ and $\overline{b}_n$. As discussed earlier, for any parameterization, it is then easy to derive $\overline{b}_1$. To simplify the exposition, it is assumed that $\underline{v}_1 = \underline{v}_n = 0$. However, the proof is easily modified to handle $\underline{v}_1 \neq \underline{v}_n$. A reserve price can also be accommodated.

For ease of notation in formulating the result, let $m = m_1 + m_n - 1 = n - 1$ denote the number of rivals faced by any bidder. Finally, define $\kappa(m_1, m_n)$ and $\tau(m_1, m_n)$, respectively, as

$$\kappa(m_1, m_n) = \frac{(m_1 + 1)\, m - \sqrt{(m_1 + 1)^2\, m^2 - 4m_n m_1 m}}{2m_n},$$

$$\tau(m_1, m_n) = \frac{m_1 - \kappa(m_1, m_n)}{m_1 - 1},$$

and note that $\kappa(m_1, m_n) \in (0, 1)$ while $\tau(m_1, m_n) > 1$. Of course, both functions are independent of $\overline{v}_1$ and $\overline{v}_n$. It can be shown that $\kappa(m_1, m_n)$ is strictly increasing in both its arguments and that $\tau(m_1, m_n)$ is strictly decreasing in both its arguments.

The functions $\kappa(m_1, m_n)$ and $\tau(m_1, m_n)$ allow a complete characterization of the equilibrium $(\widehat{v}, \overline{b}_n)$ pair. Hence, it is possible to identify precisely when bid bifurcation occurs. To put this result in context, Vickrey (1961) first considered a specific two-bidder example involving asymmetric uniform distributions.[7] A half-century later, Kaplan and Zamir (2012) obtained a full, closed-form, equilibrium characterization that holds in any *two-bidder* first-price auction in which both distributions are uniform, allowing for arbitrary supports. The time gap between the two papers illustrates the magnitude of the difficulties involved in analyzing asymmetric auctions. To our knowledge, the next result is the first to address asymmetric auctions with

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[7]In fact, Vickrey (1961) assumed that one bidder's type is known, or that his type-distribution is degenerate.

more than two bidders in the uniform-distribution setting. A closed-form equilibrium characterization is not currently within reach, and may or may not be theoretically possible.

Proposition 2 *Assume $F_i(v) = \frac{v}{\overline{v}_i}$, $v \in [0, \overline{v}_i]$, $i = 1, n$, with $\overline{v}_1 > \overline{v}_n > 0$. Assume $m_1 \geq 2$, $m_n \geq 1$. Equilibrium properties depend on the relative difference between supports, $\frac{\overline{v}_1}{\overline{v}_n}$:*

1. *If $\frac{\overline{v}_1}{\overline{v}_n} \leq \tau(m_1, m_n)$ (the supports do not differ too much), then both kinds of bidders share the same maximum bid, $\overline{b}_n = \overline{b}_1$ and $\widehat{v} = \overline{v}_1$, with*

$$\overline{b}_n = \frac{m}{\overline{v}_1 m_n + \overline{v}_n m_1} \overline{v}_1 \overline{v}_n.$$

2. *If $\frac{\overline{v}_1}{\overline{v}_n} > \tau(m_1, m_n)$ (the supports differ considerably), then bid bifurcation occurs, $\overline{b}_n < \overline{b}_1$ and $\widehat{v} < \overline{v}_1$, with*

$$\overline{b}_n = \kappa(m_1, m_n)\overline{v}_n$$

and

$$\widehat{v} = \tau(m_1, m_n)\overline{v}_n.$$

Moreover, the equilibrium is continuous in all the parameters, m_1, m_n, $\overline{v}_1$ and $\overline{v}_n$.

Proof. See the Appendix. ∎

Proposition 2 refines Corollary 1 in a special case.

In Figure 2, we consider bidders who draw valuations from uniform distributions for which we consider various combinations of $(\overline{v}_i, \overline{v}_j)$. The diagonal line from the southwest corner to the northeast corner is the symmetric bidder case ($\overline{v}_i = \overline{v}_j$). Northwest (southeast) of this line, bidders of class j (i) are the strong bidders as the highest possible valuation for these bidders exceeds that of bidders from the rival class. The various lines that are plotted correspond to the sufficient condition for bid bifurcation from Corollary 1 (which applies to any distribution) and the necessary condition from Proposition 2 (which is uniform-distribution-specific). In the white areas, the sufficient condition for bid bifurcation from Corollary 1 and the necessary condition from Proposition 2 are both satisfied and bid bifurcation occurs. In the darker shaded areas, the condition specified in Corollary 1 is not met, but bid bifurcation still occurs because the condition from Proposition 2 holds. Only for the
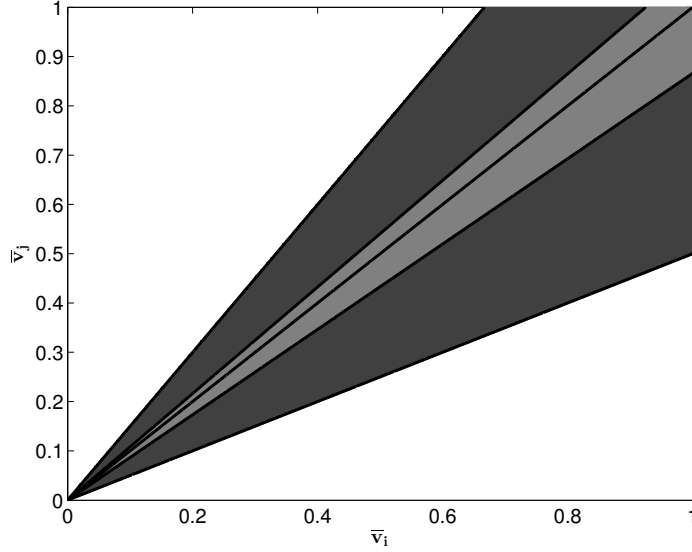
Figure 2: Bid Bifurcation with $\underline{v}_1 = \underline{v}_n = 0$

lightly-shaded gray, cone-shaped area (partitioned by the line $\overline{v}_i = \overline{v}_j$) do all bidders tender the same high bid and bid bifurcation will not occur. Note that the regions themselves are asymmetric—this is because rather than reflect the same example over the $\overline{v}_i = \overline{v}_j$ line, we always consider the case where $(m_i, m_j) = (2, 3)$ so that when bidders of class j are strong, there are three strong and two weak bidders, while the opposite is true when bidders of class i are strong. The figure helps visualize, within the context of examples concerning the uniform distribution, the magnitudes of the light gray region in which bid bifurcation does not occur (as researchers have most commonly assumed) as well as the region in which the sufficient condition for bid bifurcation from Corollary 1 is not met, yet bid bifurcation still obtains.

The next two corollaries describe how the equilibrium changes when one of the two groups of bidders become "stronger." The weak group, for instance, becomes stronger if either $\overline{v}_n$ or m_n increase. Interestingly, the comparative statics are not only quantitatively but also qualitatively different.

Corollary 3 (The weak group becomes stronger) *In the uniform model in Proposition 2, (1) $\overline{b}_n$ is strictly increasing in $\overline{v}_n$, whereas $\widehat{v}$ is weakly increasing in $\overline{v}_n$, and (2) $\overline{b}_n$ is strictly increasing in m_n, whereas $\widehat{v}$ is weakly decreasing in m_n.*[8]

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[8]The proof of the corollary establishes that as long as $\widehat{v} < \overline{v}_1$, $\widehat{v}$ is in fact strictly increasing in $\overline{v}_n$ and strictly decreasing in m_n.

Proof. See the Appendix ∎

The first part of Corollary 3 is intuitive. As $\overline{v}_n$ increases, the weak group is reinforced with types that have a higher willingness-to-pay. The presence of these new types puts upwards pressure on $\overline{b}_n$. To understand how $\widehat{v}$ changes with $\overline{v}_n$, it is useful to consider the extreme cases. If $\overline{v}_n$ is close to zero, the weak group presents no threat to the strong group, and $\widehat{v}$ will itself be small. On the other hand, if $\overline{v}_n$ is close to $\overline{v}_1$, the asymmetry between the two groups almost disappears. In this case, $\widehat{v}$ must be close to $\overline{v}_n$.

The second part of the corollary is more interesting. It is hardly surprising that adding bidders to the auction intensifies competition and drives up $\overline{b}_n$. It is more interesting to note that $\widehat{v}$ decreases as m_n increases. That is, more types among the strong bidders will "break off" from the weak group, even though they must bid higher to do so (as $\overline{b}_n$ increases). An important implication is that *bid bifurcation is more likely to occur the bigger the weak group is.* To understand the intuition, consider the limit as $m_n \rightarrow \infty$. Competition is so intense among the weak group that these bidders will bid close to their true type. The strong bidders are concerned with the highest type among the weak bidders. As $m_n \rightarrow \infty$, this type approaches $\overline{v}_n$. Thus, from the strong bidders' point of view, $\overline{v}_n$ eventually takes a role similar to a reserve price; a bid below $\overline{v}_n$ results in a loss with near-certainty. Consequently, $\widehat{v}$ must decrease, and eventually approach $\overline{v}_n$.

Corollary 4 (The strong group becomes stronger) *In the uniform model in Proposition 2, (1) $\overline{b}_n$ and $\widehat{v}$ are strictly increasing in $\overline{v}_1$ as long as $\overline{v}_1 \leq \tau(m_1, m_n)\overline{v}_n$ (i.e., $\widehat{v} = \overline{v}_1$) and independent of $\overline{v}_1$ thereafter, and (2) $\overline{b}_n$ is strictly increasing in m_1, whereas $\widehat{v}$ is weakly decreasing in m_1.*

Proof. See the Appendix. ∎

Corollary 4 confirms the intuition that weaker bidders are forced to become more aggressive, meaning that $\overline{b}_n$ increases, when they face tougher competition. The first part of the corollary reveals that once bid bifurcation occurs, it will occur at the exact same $\widehat{v}$ cut-off type even as $\overline{v}_1$ increases further. This is because the system in (2) does not change when $F_1(v)$ changes from $\frac{v}{\overline{v}_1}$ to $\frac{v}{\overline{v}'_1}$, $\overline{v}'_1 > \overline{v}_1 > \widehat{v}$. In fact, this argument is true whenever F_1 is "stretched", changing from $F_1(v)$ to $\lambda F_1(v)$ on the relevant subset of the support, $v \in [\underline{b}, \widehat{v}]$, $\lambda \in (0, 1)$. In the general case, however, a perturbation of $F_1(v)$ should be expected to affect the equilibrium value of $(\widehat{v}, \overline{b}_n)$.
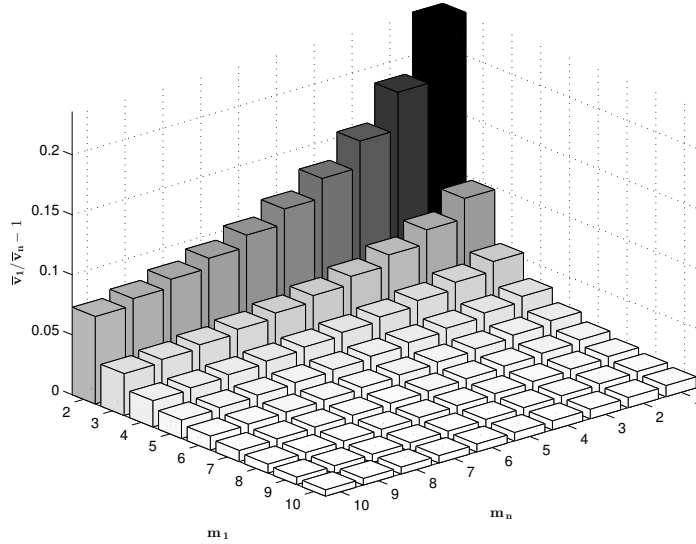
Figure 3: Prevalence of Bid Bifurcation given Bidder Composition

Finally, $\widehat{v}$ is weakly decreasing in m_1 because the more intense competition within the strong group spurs more aggressive bidding behavior, such that more strong types separate away from the weak group. In fact, since $\tau(m_1, m_n)$ is strictly decreasing in m_1, *bid bifurcation is more likely to occur the bigger the strong group is.*

The second parts of Corollary 3 and Corollary 4 together imply that as the number of bidders increases, it is more likely that bidders do not share a common maximum bid. In Figure 3, we convey this for the case of asymmetric uniform distributions. The three-dimensional bar graph plots m_1 and m_n along the horizontal axes with $(\overline{v}_1/\overline{v}_n - 1)$ on the vertical axis. Specifically, the height of the bars indicate the maximum percentage that $\overline{v}_1$ can exceed $\overline{v}_n$ without bid bifurcation occurring. As an example, the largest bar represents the most likely scenario for a common bid support—when $(m_1, m_n) = (2, 1)$ in which case if v_1 exceeds v_n by more than 23.61%, bid bifurcation must occur. Notice that regardless of whether there is an increase in the number of weak or strong bidders (or both), the height of the bars decrease rapidly. For instance, if (m_1, m_n) were instead $(3, 2)$, only a 8.11% difference between $\overline{v}_1$ and $\overline{v}_n$ would imply bid bifurcation.[9] Holding the number of strong bidders fixed,

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[9]This partition of the five bidders is not unreasonable. For example, Krasnokutskaya and Seim (2011) report that on average there were 2.6 large bidders and 1.7 small bidders vying for contracts in their California procurement data. Moreover, on average 6.6 large firms and 3.9 small firms requested plans for the contracts which is typically the proxy empirical researchers use for the

the threshold difference falls slower as m_n increases than in the opposite case where the number of weak bidders is fixed and m_1 increases. There are 90 bars depicted and 74 of them take on a value less than 0.05, meaning if the strong bidders' high type exceeds the weak bidders' strong type by five percent, bid bifurcation will occur for all of those instances. If the magnitude considered is 0.02, 52 of the instances still result in bid bifurcation.

The figure helps convey the practical importance of the corollaries discussed for the uniform model. Recall the argument in the introduction that preferential treatment of one group of bidders is mathematically equivalent to rescaling the support. The uniform example thus demonstrates that even if the two groups are initially identical, even a modest preference rate could lead to bid bifurcation—meaning simulations, counterfactual experiments, and even structural estimation routines cannot be based on an assumption of a common maximal bid. Lastly, notice one implication of Corollary 1 is that in a general model, bid bifurcation must occur if $\frac{\overline{v}_1}{\overline{v}_n} > \frac{m_1}{m_1-1}$. If $m_1 = 2$, for instance, it is thus sufficient that $\overline{v}_1$ is twice as large as $\overline{v}_n$ (assuming $\underline{b} = 0$). The larger m_1 is (or the larger $\underline{b}$ is), the lower the relative difference between the high types of each group in order to guarantee bid bifurcation. That said, these relative differences might be smaller for bid bifurcation to occur as the uniform example suggested—recall that the dark shaded areas in Figure 2 represent instances in which the high types were quite close, but the sufficient condition was not satisfied.

3.2 Comparative statics for the general case

In the previous subsection, we focused on examples involving uniform distributions. As such, the shape of the distributions were held constant but the supports and number of bidders were allowed to vary. We consider the opposite in this subsection in which we provide some results for the general case. Thus, fix the size of the two groups, $m_1, m_n \geq 2$ and the supports $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_n, \overline{v}_n]$, respectively. We consider the consequences of changing the pair of distributions from (F_1, F_n) to (G_1, G_n). Recall that G_i (strictly) dominates F_i in terms of the reverse hazard rate if

$$\frac{g_i(v)}{G_i(v)} > \frac{f_i(v)}{F_i(v)} \text{ for all } v \in (\underline{v}_i, \overline{v}_i].$$

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number of potential bidders.

Borrowing from Lebrun (1998), we will write $G_i \succ F_i$ if the above holds. Here, we will assume that G_i either strictly dominates F_i in terms of the reverse hazard rate or is identical to F_i. Borrowing from Lebrun (1998) once again, this will be denoted $G_i \succeq F_i$.

There are substantial technical difficulties involved in analyzing asymmetric first-price auctions in general, some of which come into play in the current comparative statics exercise as well. Lebrun (2006, footnote 8) describes the potential pitfalls stemming from the fact that $\ln F_i(\varphi_i(b))$ tends to $-\infty$ if b tends to $\underline{v}_i$. Lebrun (2006) cleverly solves these issues, at the cost of a somewhat more complicated proof technique. Here, we settle for the simpler proof that can be constructed with the assumption that there is a binding reserve price (r) in place, with $r \in (\underline{v}, \min\{\overline{v}_1, \overline{v}_n\})$. This assumption greatly simplifies the proof and seems conceptually to come at little cost, since r could be arbitrarily close to $\underline{v}$. The system of differential equations must now satisfy $\varphi_i(r) = r$, $i = 1, n$ (Lebrun (2006)). The technical significance of the assumption is that $\ln F_i(\varphi_i(b))$ is finite even as $b \to r$. Lebrun's (2006, Section 3) simpler proof technique can then be adapted to prove the main result in the current subsection.

Let $(\widehat{v}^F, \overline{b}_n^F)$ and $(\widehat{v}^G, \overline{b}_n^G)$ denote the equilibrium values of $(\widehat{v}, \overline{b}_n)$ when the distributions are (F_1, F_n) and (G_1, G_n), respectively. The characterization in Proposition 1 implies that $\widehat{v}$ is non-increasing with $\overline{b}_n$.

Proposition 3 *Assume $G_i \succeq F_i$, $i = 1, n$ and $m_1, m_n \geq 2$. Assume there is a binding reserve price in place, with $r \in (\underline{v}, \min\{\overline{v}_1, \overline{v}_n\})$. Then, $\overline{b}_n^G \geq \overline{b}_n^F$ and thus $\widehat{v}^G \leq \widehat{v}^F$. Consequently, if bid bifurcation occurs under (F_1, F_n) it also occurs under (G_1, G_n); i.e., when bidders become stronger.*

Proof. See the Appendix. ∎

Proposition 3 is consistent with intuitive comparative statics in Lebrun (1998). With two groups of bidders who share the same support, Lebrun (1998) shows that the common maximum bid must increase when one group of bidders become stronger. Applied to our setting, this would suggest that if we hold $\widehat{v}$ fixed at $\widehat{v}^F$, then $\overline{b}_n$ should increase when (F_1, F_n) is replaced by (G_1, G_n). Given the inverse relationship between $\overline{b}_n$ and $\widehat{v}$ (see Proposition 1 or the piecewise linear functions in Figure 1), it is then no surprise that the latter decreases.

It is useful to compare, or rather combine, Propositions 2 and 3. Note that any convex distribution function on $[0, \overline{v}_i]$ strictly dominates the uniform distribution on $[0, \overline{v}_i]$ in terms of the reverse hazard. Proposition 3 thus implies that if the distributions are convex, then bid bifurcation is more likely to occur (i.e., occurs for more $(\overline{v}_1, \overline{v}_n)$ pairs) than in the uniform benchmark. Letting r approach zero then allows Proposition 2 to be used as a lower bound on the incidence of bid bifurcation.

4 Solving for asymmetric equilibrium

As we are the first to delve into the general asymmetric model with more than two bidders, researchers who have solved asymmetric auctions have typically been careful to limit attention to settings in which the standard boundary condition is sure to hold. Li and Riley (2007, footnote 8) explicitly note that they restrict attention to the case of identical supports. Instead, they argue that if the supports differ one could replace the distributions with approximations which do have a common support. Likewise, Hubbard and Paarsch (2014) claimed that boundary conditions might need adjustment in settings where the type supports differ and there are more than two bidders at auction, and so they explicitly restricted their analysis to the case of common type supports in trying to avoid such complications. These researchers, however, are among the few to explicitly recognize that bid supports may differ. That said, no one has proposed a solution technique that solves for the equilibrium of an asymmetric auction when bid bifurcation is possible.

As we have argued above, casual observation of auctions suggest bid bifurcation is a likely possibility in some settings and empirical researchers encounter raw data that suggests bidding supports may differ. Moreover, allowing for bid bifurcation is required to evaluate real-world policies such as bid preferences, where any avoidance of the bid bifurcation issue (in the spirit of the proposal by Li and Riley) would require approximations to the type distributions which would need to vary with the preference rate. Given the potential for bid bifurcation in real world auctions, and the disadvantages that come with using approximations which map type distributions over different supports to ones on a common support in order to avoid the issue, we seek to strengthen the approach to solving an asymmetric auction. There are no existing solution techniques that can accommodate settings as those we have suggested. In the first subsection, we provide a way for solving a general asymmetric auction that allows

for bid bifurcation. We then complement this proposal with some solved examples in the second subsection—first demonstrating the ability of our approach to determine equilibrium in examples we know much about from the insights documented above (such as ones involving uniform distributions) before venturing into situations where little is known a priori.

4.1　General solution approach

Unlike the symmetric first-price model which admits a closed-form solution for the Bayes–Nash equilibrium bid functions, asymmetric auctions rarely allow for such tractability. As such, researchers are left resorting to numerical methods to solve the system of differential equations under the appropriate boundary conditions. Hubbard and Paarsch (2014) summarize the various ways in which researchers have gone about solving for the equilibrium which can be broadly categorized as: (i) shooting algorithms, (ii) fixed-point iterations, and (iii) a polynomial approach. Rather than delve into the merits of each of these approaches, we first describe a general strategy and then provide a discussion of important steps.

1. Compute $b^c = m_1 \overline{v}_n - (m_1 - 1)\overline{v}_1$.

2. Check whether $b^c < \underline{b}$, if so bid bifurcation must occur as the sufficient condition from Corollary 1 holds; if not, bid bifurcation may still occur as the condition in Corollary 1 is sufficient, but not necessary.

3. Compute the maximum bid tendered in a symmetric auction amongst m_1 class 1 bidders. If this bid is greater than b^c, bid bifurcation must happen given Corollary 2 which, again is sufficient but not necessary.

4. If one of the previous two steps did not suggest bid bifurcation, there is one more way in which b^c can be useful. Importantly, b^c provides an upper-bound on $\overline{b}$ when the bid support is common to bidders and a lower-bound on $\overline{b}_n$ when bid bifurcation obtains. Because of this, b^c can be used in a diagnostic test which allows for detecting whether bid bifurcation happens or not. Specifically, consider imposing the boundary conditions $\varphi_i(b^c) = \overline{v}_i$, $i = 1, n$, and integrating the system (2) backwards under the assumption that the equilibrium involves a common bid support. One of two things will happen in integrating backwards:

(a) The system blows up (becomes unbounded, going off to negative infinity) in approaching $\underline{b}$. We argue (see below) this is evidence that all bids are below b^c in equilibrium—in particular, $\bar{b} \leq b^c$ (bidders share a common bid support);

(b) The inverse bid functions return values in the range $[\underline{v}, \bar{v}_n]$. We argue (see below) this is evidence that some bidders tender amounts that exceed b^c in equilibrium—in particular, $\bar{b}_n, \bar{b}_1 \geq b^c$ (bid bifurcation occurs).

5. Given the outcome of the previous steps, if the equilibrium involves bid bifurcation, solve the system (2) on the overlapping region of the bid support and then integrate (4) over the region that only class 1 bidders are active. If the equilibrium involves a common bid support, solve the system (2) over the entire support.

The first three steps follow directly from the theory presented and discussed in earlier sections of the paper. The fourth step warrants some further discussion given it allows us to understand whether bid bifurcation occurs should the theoretical conditions from Corollary 1 and 2 fail. Recall the panels of Figure 1 presented earlier and note that if any bids (in particular the bids submitting by high types) exceed b^c, it must be that $\hat{v} < \bar{v}_1$ and, by Proposition 1, $\hat{v}$ is pinned down by the downward sloping line in the panels. Similarly, if bids are all below b^c, then $\hat{v} = \bar{v}_1$ and bidders share a common bid support. The test allows us to decipher whether bid bifurcation occurs or not because the equilibrium solution is monotonic in $\bar{b}$—something critical to researchers who adopt shooting algorithms as such behavior guides guesses for initial conditions across iterations; see, for example, Marshall, Meurer, Richard, and Stromquist (1994). Fibich and Gavish (2011) showed that the shooting algorithm is inherently unstable and the comparisons in Hubbard and Paarsch (2014) echo that this approach has issues.[10] As such, we are not advocating an attempt to solve for the equilibrium by integrating backwards which would involve shooting as a basis for search of $\bar{b}_n$. We integrate backwards (once) not to actually approximate equilibrium, but to see whether the system blows up when starting from b^c or not.[11] That is, it

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[10]To be clear, the instability documented relates to shooting backwards from an ε-neighborhood of the true $\bar{b}$. In our case, we are interested in investigating behavior of the system when integrating backwards from the point b^c.

[11]As an aside, there may be a way to modify the fixed-point iteration approach of Fibich and Gavish (2011) so that it is capable of diagnosing whether bid bifurcation occurs or not. It would

is precisely the tendency of the solution to blow up when fed a high bid that is too high that guides our diagnostic test.

To consider the accuracy of this detection step, consider the plots depicted in Figure 4. In the left panel, we consider asymmetric uniform auctions in which $(m_1, m_n) = (3, 2)$ where we vary $\overline{v}_n$ in the plot. Given a value for $\overline{v}_n$ (and $\overline{v}_1$ which is set equal to one), we compute b^c and integrate the system backwards. We record whether step 4 above suggests bid bifurcation or not and define an indicator variable, referred to as "Terminal Indicator" in the figure, that equals one if the diagnostic test in step 4 suggests bid bifurcation, and zero if it suggests the bid support is common for all bidders. We also report threshold values implied by Corollary 1 and Corollary 2. Specifically, for all points to the left of "Cor 1 Bifurcation Indicator", the condition of Corollary 1 is satisfied. For all points the left of "Cor 2 Bifurcation Indicator", the condition of Corollary 2 is satisfied. (Note that, here $\underline{b} = 0$ so Corollary 2 is stronger than Corollary 1.) In addition, we denote whether the necessary and sufficient condition of Proposition 2 is satisfied. For every value of b^c implied by the given $\overline{v}_n$, the prediction of step 4 matched the prescription of Proposition 2.[12]

In the middle panel, we consider a situation in which bidders draw types from truncated normal distributions. Again, consider $(m_1, m_n) = (3, 2)$ and fix $\overline{v}_1 = 1$ while letting $\overline{v}_n$ vary. In this example, we also fix the parameters of group 1's distribution to be a normal distribution with mean 0.5 and standard deviation 0.25 which is truncated over the [0,1] support. For every $\overline{v}_n$, we consider the group receives valuations from a normal distribution with mean $\mu_n = \overline{v}_n/2$, has a fixed standard deviation, and is truncated over $[0, \overline{v}_n]$. In this figure, we indicate the threshold point at which the Terminal Indicator switched from predicting bid bifurcation to predicting a common support for three parameterizations, corresponding to different values of the variance for group n's distribution. The condition of Corollary 1 is independent of the parameters of the distributions, depending only on the supports and number of players. As such, this threshold is fixed as group n's distribution changes. Because we have fixed the distribution of group 1, Corollary 2 is also fixed across examples. Note that, although we did not require it in this exercise, the step 4 test always predicts

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seem that one would need to embed flexibility in the boundary condition on the tying function, $t(v)$, that describes the type of bidder 1 that ties with (bids the same as) bidder n with type v to allow for the possibility that bidding supports may differ—so $t(\overline{v}_n)$ need not equal $\overline{v}_1$.
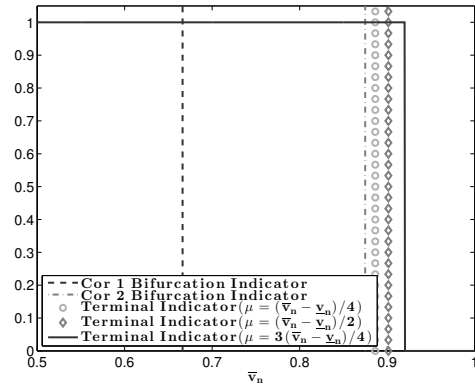
[12]In constructing this figure, we consider a grid of $\overline{v}_n$ values in which points are 0.0005 apart from each other.
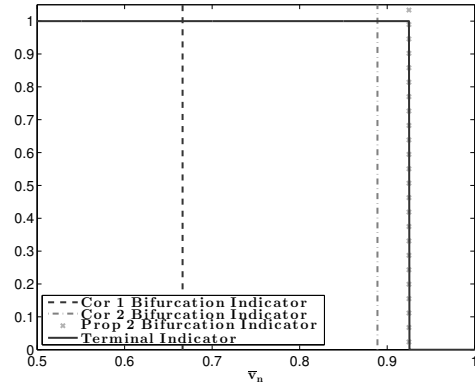
(a) Uniform Example
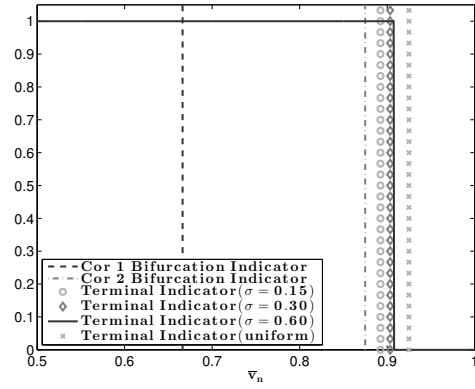
(b) Truncated Normal (σ_n)

(c) Truncated Normal (μ_n)

Figure 4: Predictions of Diagnostic Test for Bid Bifurcation

bid bifurcation when these sufficient conditions are met. Moreover, the predictions change in reassuring ways—when the σ_n is low, the terminal indicator approaches the Corollary 2 threshold while, when the σ_n is high the terminal indicator approaches the uniform distribution threshold. When group n's distribution has a lower variance, high types of group 1 need not worry as much about facing competitive group n bidders for a given $\overline{v}_n$. As such, competition within group n is what disciplines bidding behavior in a way that is similar to the symmetric auction setting involving only m_1 bidders. Likewise, when the variance of the group n distribution is high, the density becomes very flat, mimicking the uniform distribution. Though the comparison to the uniform case is not valid (remember that group 1's distribution is a fixed truncated normal distribution), we see that bid bifurcation approaches the uniform threshold. In fact, if the standard deviation of both groups is set sufficiently high, the predictions of the diagnostic test are exactly the same as in the uniform distribution case—the Terminal Indicator line falls directly on the Proposition 2 indicator depicted to the far right in the middle panel figure.

Lastly, consider the panel on the right. In this case, we again consider $(m_1, m_n) = (3, 2)$ bidders with types drawn from asymmetric truncated normal distributions over $[0, \overline{v}_i]$ where $\overline{v}_n$ changes in the figure. Group 1's distribution is fixed to be a normal distribution with mean 0.5 and standard deviation 0.25 which is truncated over the $[0, 1]$ support. In this figure, $\sigma_n = 0.25$ for all parameterizations but we let μ_n vary. Specifically, we depict situations in which $\mu_n = \delta(\overline{v}_n - \underline{v}_n)$ where $\delta = \{1/4, 1/2, 3/4\}$. For a given $\overline{v}_n$, since σ_n is fixed, higher values for δ imply distributions that first-order stochastically dominate distributions associated with lower values of δ. Thus, by construction, we can investigate whether the diagnostic test respects the predictions of Proposition 3. Note that the figure in the right panel identifies the same relationship as this theoretical result—higher values of δ imply bid bifurcation occurs more often. Said another way, for a given $\overline{v}_n$, if bid bifurcation holds for a low value of δ, it is also true that bid bifurcation occurs for a higher value of δ.

With confidence that the type of equilibrium can be deciphered from this diagnostic test, we now turn to the final step of approximating the equilibrium. We should note that our goal is not to take a stance on which solution technique is the best, but rather to make clear how the theory presented above can guide and modify the

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implementation of such algorithms.[13] If bidding is over a common support, the structure of the problem is exactly the same as the standard setting—though note, we can use the insight above to inform a better initial guess for $\bar{b}$ which we know must be in the range $[\underline{b}, b^c]$.[14] On the other hand, if bid bifurcation occurs, the key input to our algorithm will be a guess on $\bar{b}_n \in [b^c, \overline{v}_n]$, paired with the theoretically consistent $\widehat{v}$ from Proposition 1, to approximate the system (2) over the intersection of the bid supports.

Here, we describe a solution technique which builds off the polynomial-based approach considered in Hubbard, Kirkegaard, and Paarsch (2013). Under this approach, the inverse bidding strategies are assumed to be polynomials, the coefficients of which are chosen to solve approximately the system of differential equations that characterize equilibrium behavior along with the unknown high bid such that conditions provided by theory (such as the appropriate boundary conditions) hold. Thus, assuming the equilibrium strategies can be approximated by Chebyshev polynomials, the inverse bid function for bidder i can be expressed as

$$\hat{\varphi}_i(b; \boldsymbol{\alpha}_i, \bar{b}_n) = \sum_{k=0}^{K} \alpha_{i,k} T_k[x(b; \bar{b}_n)] \quad i = 1, n \tag{10}$$

where $x(\cdot)$ lies in the interval $[-1, 1]$ and where, for completeness, we have explicitly defined it as a transformation of the bid b under consideration.[15] Here, $T_k(\cdot)$ denotes the kth Chebyshev polynomial of the first kind and the vector $\boldsymbol{\alpha}_i$ collects the

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[13]For example, building on our comments in footnote 11, it may be possible to modify the fixed-point approach of Fibich and Gavish (2011) to handle bid bifurcation as well—the boundary conditions would need to be adjusted appropriately. For example, one could replace $\overline{v}_1$ in the boundary condition $t(\overline{v}_n) = \overline{v}_1$ with some $\widehat{v} < \overline{v}_1$ guess, and then use the fixed-point algorithm to derive a candidate solution. If the solution does not return the value $\bar{b}_n$ in Proposition 1, the $\widehat{v}$ guess is incorrect. One could then iterate on this procedure until a theoretically-consistent $(\bar{b}_n, \widehat{v})$ pair is reached. Unfortunately, because the fixed-point approach involves a transformation it suppresses the one item, $\bar{b}_n$, about which we have some information (Proposition 1) which requires an iterative scheme to be written around the core fixed-point algorithm. In contrast, the approach we describe builds the $\bar{b}_n$-$\widehat{v}$ relationship directly into the algorithm.

[14]In the standard setting, the only information known a priori is that $\bar{b} \in [\underline{b}, \overline{v}_n]$.

[15]We abuse notation slightly so that, in the case where bidders tender offers over a common support, $\bar{b}_n = \bar{b}$.

polynomial coefficients for bidder i. Thus,

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \quad k = 1, 2, \ldots, K - 1.$$

We evaluate the system (2) over the Chebyshev nodes from the relevant bid support which have the property of minimizing the maximum interpolation error when approximating a function and so are often considered a good choice for the requisite grid. Casting the problem within the Mathematical Programming with Equilibrium Constraints (MPEC) approach proposed by Luo, Pang, and Daniel (1996) and advocated by Su and Judd (2012), the unknowns $(\boldsymbol{\alpha}_i, \bar{b}_n)$ are then chosen to minimize the deviations from the system (2) at the grid points such that the relevant theoretical (importantly, the boundary) conditions hold. Our focus has been on how the boundary conditions are modified if bidding supports differ. Once the boundary conditions are modified, our approach mirrors that of Hubbard, Kirkegaard, and Paarsch (2013) where readers can find an extended discussion of the shape-based constraints that guide search for the polynomial coefficients and high bid. Because Proposition 1 reduces the dimensionality of the problem so that instances involving bid bifurcation are no more complex than the common support setting, the approach generalizes naturally once the boundary conditions are modified with the exception that, under bid bifurcation, the inverse bid function of group n must satisfy the derivative condition of our Lemma 2 above.

4.2   Solved examples

The theory we present above is most precise for situations in which bidders draw valuations from asymmetric uniform distributions. Continuing the tradition of Vickrey (1961) and Kaplan and Zamir (2012), we first consider two examples involving bidders who receive types from uniform distributions where, because more than two bidders are considered, no closed-form solution exists.

EXAMPLE 1: Consider an auction with $m_1 = 3$ group 1 bidders who draw valuations from a uniform distribution over the support $[0, 1]$ and $m_n = 2$ group 2 bidders who draw valuations from a uniform distribution over the support $[0, 3/4]$. In this setting,

$b^c = 1/4$. If the m_1 bidders were at a symmetric auction, the equilibrium strategy would prescribe high-type bidders tender a bid of $\overline{b}^s = 2/3$. Thus, the condition of Corollary 2 holds and we know bid bifurcation will occur. Moreover, applying what we know from Proposition 2(b): $\overline{v}_1/\overline{v}_n = 4/3 > 1.0811 = \tau$ so the necessary and sufficient condition which applies to this example confirms that bid bifurcation must happen. We present the inverse bid functions approximated by following the algorithm proposed above for this example in Figure 5(a). Note that the group n inverse bid function satisfies the property of Lemma 2 as it flattens out so that its (left-)derivative is zero at $\overline{b}_n$. The line indicating the value for $\widehat{v}$ suggests that for about 19% of group 1 types, they compete only with other group 1 bidders. The inverse bid functions are nearly identical for low bids but as they approach $\overline{b}_n$ there is a clear divergence.

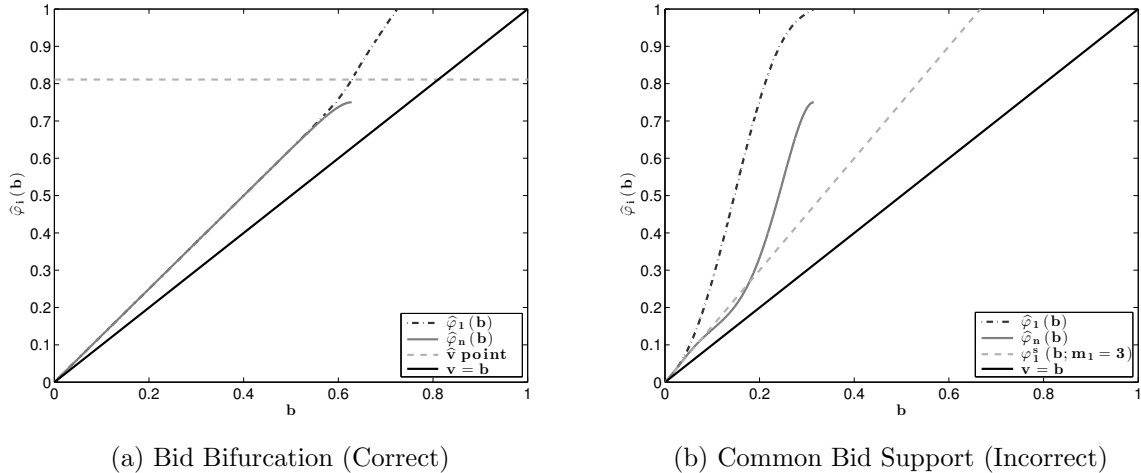(a) Bid Bifurcation (Correct)   (b) Common Bid Support (Incorrect)

Figure 5: EXAMPLE 1: Imposing Bid Bifurcation vs. Common Bid Support

For Example 1, we complement the correct solution with an illustration of how things look if the wrong boundary condition is imposed. Specifically, in the panel on the right we present the output of an approximation in which the common bid support conditions are imposed. There is a stark contrast across these figures and the wrong solution on the right should raise some immediate flags. Recall that a symmetric auction with m_1 bidders leads to high types tendering a bid of $\overline{b}^s = 2/3$; in fact the symmetric auction equilibrium involves a linear bid function with all types tendering bids that are 2/3 of their valuation. When m_n bidders enter the auction, the

presence of the m_n bidders means a group 1 bidder should behave more aggressively (or at least no less competitively) given the presence of the same $(m_1 - 1)$ rivals and the m_n bidders. This behavior propagates through to the group 1 bidders with types that exceed the highest group n type given they now face more aggressive behavior from their rivals with lower valuations. The symmetric auction inverse bid function involving $m_1 = 3$ bidders is depicted as the lighter, dashed line which we have labeled $\varphi_1^s(b; m_1 = 3)$. More aggressive behavior would mean that the $\varphi_1(b)$ function in the asymmetric auction should lie between this linear function and the 45° line. In fact, the exact opposite is happening in the figure as bidders shade their valuations by even more.[16] Were one to assume this was the equilibrium as opposed to the (correct) one in the panel on the left involving bid bifurcation, any inference based off these approximations would suggest drastically different implications. For example, consider ranking the first-price auction relative to a second-price auction according to expected revenues. The correct solution suggests slightly higher revenues can be expected from a first-price auction, while the incorrect solution predicts expected revenues from a first-price auction that are less than half of the amount that can be expected from a second-price auction.[17] Thus, the incorrect solution not only substantially miscalculates expected revenue, but it would lead to the wrong revenue ranking as well.

EXAMPLE 2: Consider an auction with $m_1 = 3$ group 1 bidders who draw valuations from a uniform distribution over the support $[0, 1]$ and $m_n = 2$ group 2 bidders who draw valuations from a uniform distribution over the support $[0, 0.95]$. In this setting, $b^c = 0.85$. If the m_1 bidders were at a symmetric auction, the equilibrium strategy would again prescribe bidders tender 2/3 of their valuations since their distribution did not change from Example 1. Importantly, because now $\bar{b}^s < b^c$, the condition of Corollary 2 is not met (again, nor is the Corollary 1 condition) meaning bidders will tender offers over a common support. Moreover, applying what we know from

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[16]Though we depict this improper approximation, we should note that if the wrong boundary conditions are imposed, the nonlinear optimization solver we use (SNOPT) cannot achieve convergence and typically reports that it cannot proceed into an undefined region. We are reassured by this as it suggests the problem is sufficiently well disciplined and users would recognize immediately any difficulties.

[17]Expected revenue from a second-price auction is 0.5949. Using the correct bid bifurcation solution, the winning bid (tendered by a group 1 bidder) exceeds $\bar{b}_n$ 47% of the time and expected revenue increases slightly to 0.5959. In contrast, expected revenue from the incorrect, common bid support approximation is 0.2539.

Proposition 2(a): $\overline{v}_1/\overline{v}_n = 1.0526 < 1.0811 = \tau$ so the necessary and sufficient condition which applies to this example confirms that bidding should happen on the same support for all types. We present the inverse bid functions approximated by following the algorithm proposed above for this example in Figure 6(a).

This time the algorithm that imposes (appropriately now) the boundary conditions for a common bid support succeeds in solving for the correct bid function. Building on the discussion above involving how behavior should compare across a m_1-symmetric auction and a (m_1, m_n)-asymmetric auction, the inverse bidding strategies suggest all bidders are now more aggressive in equilibrium. In this example the bidders are nearly identical. If all bidders were group 1 bidders, then the symmetric auction equilibrium strategy would involving bidders tendering 4/5 of their valuation. In this asymmetric setting, since group n bidders are slightly weaker than group 1 bidders, we see the common high bid is just below 4/5 and there is only a slight separation in behavior at the upper end of the bid support. Note, too, that the solution to this problem, even though bid bifurcation does not obtain, was previously unattainable by researchers solving asymmetric auctions since $m_1 \geq 2$ and the type supports are different. Rather than present another approximation using the wrong boundary conditions, we note that the discussion provided above about violating equilibrium bidding behavior when m_n bidders enter the auction holds in this setting too; that is, improperly imposing bid bifurcation fails to converge to a solution and in searching for a solution continues to stray into territory which we know is theoretically invalid because m_1 bidders shade their valuations more.

EXAMPLE 3: As a final example based on bidders who receive valuations from uniform distributions, consider an auction with $m_1 = 2$ group 1 bidders who draw valuations from a uniform distribution over the support $[0, 1]$ and $m_n = 2$ group 2 bidders who draw valuations from a uniform distribution over the support $[0, 0.15]$. The extreme difference in the highest possible valuations of bidders might best correspond with the motivating example we provided in the introduction which involved two billionaire art collectors vying for a painting at an auction with two art students. We are likely being quite generous in evaluating the competitiveness of the art students in this example—if the highest possible valuation for the billionaires is just \$1,000,000, the highest possible valuation of students is still \$150,000. Regardless, in this setting, bid bifurcation is certain—the sufficient conditions of Corollary 1 and 2 as well as the necessary and sufficient condition of Proposition 2(b) for bid bifurcation are easily

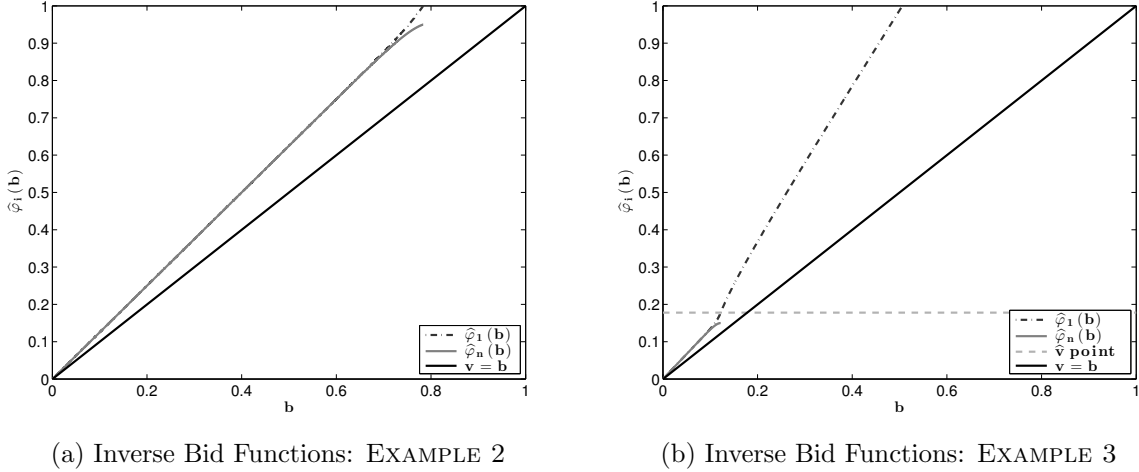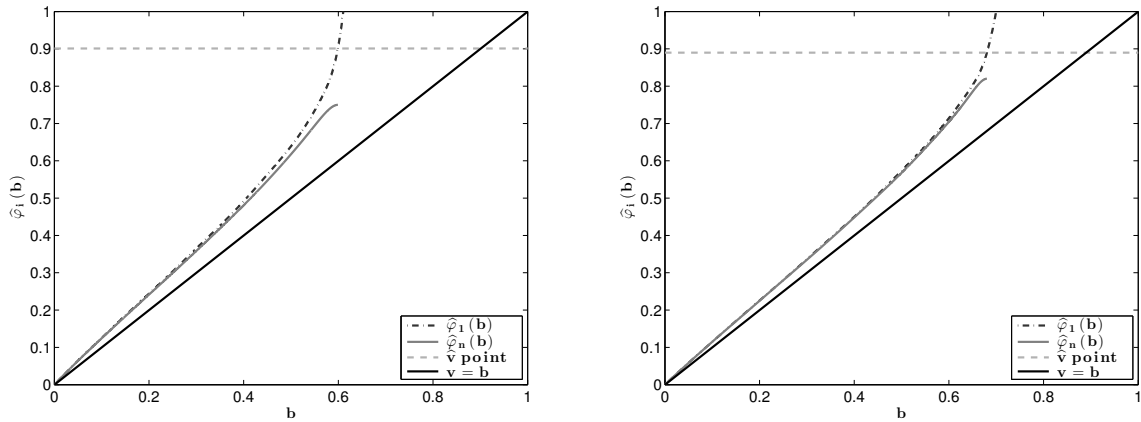(a) Inverse Bid Functions: EXAMPLE 2   (b) Inverse Bid Functions: EXAMPLE 3

Figure 6: EXAMPLES 2 & 3: Inverse Bid Functions

satisfied. Assuming a common bid support in this setting would be disastrous as, in equilibrium, less than 18% of the strong types tender bids that fall within the bid support of the weak bidders. We present the inverse bid functions approximated by following the algorithm proposed above for this example in Figure 6(b). If a common bid support were assumed, then no bid could exceed 0.15 which assumes students bid as aggressively as possible by tendering their valuations. Actually, the high bid for the strong group (0.51) is more than three times the highest type of the weak group. Since a symmetric auction with only the two billionaires would involve each tendering half of their valuation in equilibrium, the mere presence of the student bidders leads to an increase in the bids of all strong types which propagates through to the (much larger) share of the strong group's bid support for which the weaker bidders are not even active.

EXAMPLE 4: Having given some credibility to the solution approach suggested by considering examples with uniform distributions, let us venture into territory that is less convenient analytically. The general solution approach we provide is not limited in any way by the distribution considered so long as the standard conditions for uniqueness are satisfied (see the conditions we discussed at the beginning of Section 2, just before Lemma 1). Consider instead an auction with two bidders from each group at auction. Let the valuations of group 1 bidders be distributed according to a Normal distribution with mean 0.5 and standard deviation 0.25 which has been

truncated over the support $[0, 1]$. Likewise, assume group n bidders draw types which are distributed according to a Normal distribution with mean $\overline{v}_n/2$ and standard deviation 0.25, which has been truncated over the support $[0, 0.75]$. In this example, the type distribution of group 1 bidders dominates that of group n bidders in terms of the reverse hazard rate. Theory tells us this stochastic ranking of distributions is a sufficient condition for weakness to breed aggression on overlapping portions of the bid supports; for a given type, group n bidders should tender an offer that exceeds that of group 1 bidders. For this example, $b^c = 1/2$ and step 4 of the solution procedure suggests bid bifurcation occurs. Indeed, our approximated solutions, depicted in Figure 7(a) reflect this feature of the solution. Note, too, that the inverse bid function of group 1 bidders is more nonlinear than the solutions involving uniform distributions but, nonetheless, the solution is still smooth at $(\overline{b}_n, \widehat{v})$. Lastly, note that if $\overline{v}_n$ is increased from 0.75 to 0.82 (or above) the equilibrium involves all bidders tendering offers over a common bid support. We continue this thought in the next example.

(a) Inverse Bid Functions: EXAMPLE 4   (b) Inverse Bid Functions: EXAMPLE 5

Figure 7: EXAMPLES 4 & 5: Inverse Bid Functions

EXAMPLE 5: Again let valuations of group 1 bidders be distributed according to a Normal distribution with mean 0.5 and standard deviation 0.25 which has been truncated over the support $[0, 1]$. Now, let's increase the highest type of group n bidders to 0.85. Assume group n bidders draw types which are distributed according to a Normal distribution with mean $\overline{v}_n/2$ and standard deviation 0.25, which has been truncated over the support $[0, 0.85]$. At the end of the last example we noted that if

$(m_1, m_n) = (2, 2)$, bid bifurcation would not occur. This implication is in the spirit of the first part of Corollary 3 (which applied to the uniform model from Proposition 2): the weak group becomes stronger because their highest possible type increases (as does the mean of their valuation distribution which is tied to the highest type by construction) and so bid bifurcation is less likely. To revisit the implications of the second parts of Corollary 3 and 4 in an situation outside of the uniform model, suppose another bidder shows up at auction from each group so that $(m_1, m_n) = (3, 3)$. Step 4 of the general solution procedure suggests bid bifurcation occurs, consistent with the implications of Corollary 3 and 4, though in a setting for which they are not known to hold. We depict the equilibrium bid functions in Figure 7(b). Again, because reverse hazard rate dominance holds, weakness leads to aggression over the common part of the bid support. Perhaps not surprisingly, because participation has increased by 50%, all bidders behave more competitively in tendering higher bids. Bid shading decreases as does the difference in the equilibrium bid functions of the two groups so that the only real separation in behavior occurs as types increase towards $\widehat{v}$. Importantly, this example serves as a reminder that bid bifurcation depends on both the relative difference between supports $(\overline{v}_1/\overline{v}_n)$ as well as the composition of (and number of) bidders at auction (m_1, m_n). This can be important when considering any number of counterfactual simulations (perhaps after having estimated type distributions in empirical work) such as the effect of preference policies and bidder subsidies, particularly when entry is endogenous so that participation may vary with the policy considered.

5   Conclusion

This paper focuses on an empirically relevant complication that arises in first-price auctions with more than two bidders, namely bid bifurcation. Our theoretical and numerical results suggest that bid bifurcation may occur in the face of even relatively small asymmetries. Nevertheless, most existing empirical and numerical methods do not take bid bifurcation into account. A main motivation of the paper is to make applied researchers aware of this oversight and to take some initial steps toward satisfactorily incorporating considerations of bid bifurcation into both theory and practice.

It should be noted that bid bifurcation may well exist outside the particular setting

we have investigated. We have, for simplicity, concentrated on first-price auctions with exactly two groups of bidders. Allowing for more groups of bidders does not appear to make bid bifurcation any less likely. Similarly, other pay-your-bid auctions—most notably the all-pay auction—will be prone to bid bifurcation as well.

Even with an arbitrary number of groups of bidders, Lemma 1 provides some equilibrium structure. We speculate that a "recursive" version of Proposition 1 could be developed to make inferences about the maximum bids of different groups and the critical types that submit bids that coincide with the maximum bid of some other group. The details, however, are left for future research.

In many ways, the all-pay auction is analytically even more complicated than the first-price auction. Parreiras and Rubinchik (2010) prove that bidding strategies need not even be continuous when there are more than two asymmetric bidders. In a related paper, Kirkegaard (2013) proves that a bidder may become worse off if he is a member of a diverse set of bidders who are given preferential treatment in an all-pay auction with more than two bidders. Bid bifurcation simply adds to these complications. We have elected to focus on the simpler first-price auction, though this choice is primarily motivated by the empirical relevance of the auction format.

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Appendix

Proof of Lemma 1. The first part is well-known, see e.g. Lebrun (1999). Kirkegaard (2009, footnote 12) uses a "revealed preference" approach to prove the result. Here, we extend the latter method to allow $\overline{v}_i \geq \overline{v}_j$. Thus, by contradiction, assume $\overline{v}_i \geq \overline{v}_j$ but $\overline{b}_i < \overline{b}_j$. Let $P(b)$ be the distribution function of the highest bid among bidder i's and bidder j's common rivals, with $P(\overline{b}_j) \geq P(\overline{b}_i)$. Consider bidder j with type $\overline{v}_j$. In order for $\overline{b}_j$ to an equilibrium bid, there must be no incentive to deviate to $\overline{b}_i$ instead. For either bid, bidder j outbids bidder i with probability 1, so the requirement is that

$$\left(\overline{v}_j - \overline{b}_j\right) P(\overline{b}_j) \geq \left(\overline{v}_j - \overline{b}_i\right) P(\overline{b}_i)$$

or

$$\overline{v}_j \left[P(\overline{b}_j) - P(\overline{b}_i)\right] \geq \overline{b}_j P(\overline{b}_j) - \overline{b}_i P(\overline{b}_i). \tag{11}$$

Similarly, bidder i with type $\overline{v}_i$ must weakly prefer bidding $\overline{b}_i$ to $\overline{b}_j$. Since a bid of $\overline{b}_i$ causes bidder i to lose to bidder j with strictly positive probability, payoff from bidding $\overline{b}_i$ is strictly smaller than $\left(\overline{v}_i - \overline{b}_i\right) P(\overline{b}_i)$. In comparison, deviating to $\overline{b}_j$ leads bidder i to outbid bidder j with probability 1. Thus, it is necessary that

$$\left(\overline{v}_i - \overline{b}_i\right) P(\overline{b}_i) > \left(\overline{v}_i - \overline{b}_j\right) P(\overline{b}_j),$$

or

$$\overline{b}_j P(\overline{b}_j) - \overline{b}_i P(\overline{b}_i) > \overline{v}_i \left[P(\overline{b}_j) - P(\overline{b}_i)\right].$$

Since the term in the brackets is non-negative and $\overline{v}_i \geq \overline{v}_j$, the last inequality implies that

$$\overline{b}_j P(\overline{b}_j) - \overline{b}_i P(\overline{b}_i) > \overline{v}_j \left[P(\overline{b}_j) - P(\overline{b}_i)\right]. \tag{12}$$

The proof concludes by observing that (11) and (12) are contradictory. ∎

Proof of Lemma 2. Assume $\overline{b}_n < \overline{b}_1$. Consider first bidder n with type $\overline{v}_n$. Let $U_n(b)$ denote the natural logarithm of his expected payoff from bidding b, with

$$U_n'(b) = \frac{-1}{\overline{v}_n - b} + (m_n - 1) \frac{d}{db} \ln F_n(\varphi_n(b)) + m_1 \frac{d}{db} \ln F_1(\varphi_1(b)) \text{ for } b \in (\underline{v}, \overline{b}_n)$$

and

$$U'_n(b) = \frac{-1}{v_n - b} + m_1 \frac{d}{db} \ln F_1(\varphi_1(b)) \text{ for } b \in (\bar{b}_n, \bar{b}_1).$$

Let $U'_n(\bar{b}_{n\nearrow})$ and $U'_n(\bar{b}_{n\searrow})$ denote the left-derivative and right-derivative at $\bar{b}_n$, respectively. In order for bidder n to not submit higher bids than $\bar{b}_n$ it is necessary that $U'_n(\bar{b}_{n\searrow}) \leq 0$. For similar reasons, it is necessary that $U'_n(\bar{b}_{n\nearrow}) \geq 0$. However, $U'_n(\bar{b}_{n\nearrow}) > 0$ can be ruled out, because in this case types marginally below $\bar{b}_n$ would find it profitable to deviate from their equilibrium bid and instead bid $\bar{b}_n$. Thus, $U'_n(\bar{b}_{n\nearrow}) = 0$. Combining these observations yields

$$\frac{d}{db} \ln F_n(\varphi_n(\bar{b}_{n\nearrow})) \geq \frac{m_1}{m_n - 1} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\searrow})) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\nearrow})) \right]. \tag{13}$$

Consider next bidder 1 with type $\hat{v} \in (\underline{v}, \bar{b}_1)$. Let $U_1(b)$ denote the natural logarithm of his expected payoff from bidding b. The derivative, $U'_1(b)$, can be calculated in much the same manner as $U'_n(b)$, the main difference being that the composition of rival bidders is different. In equilibrium, this bidder is supposed to find a bid of $\bar{b}_n$ optimal. By the above arguments, it is thus necessary that $U'_1(\bar{b}_{n\nearrow}) = U'_1(\bar{b}_{n\searrow}) = 0$, which implies that

$$\frac{d}{db} \ln F_n(\varphi_n(\bar{b}_{n\nearrow})) = \frac{m_1 - 1}{m_n} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\searrow})) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\nearrow})) \right]. \tag{14}$$

Since φ_n is non-decreasing, the term in brackets must be non-negative. If it is strictly positive, then (13) implies

$$
\begin{aligned}
\frac{d}{db} \ln F_n(\varphi_n(\bar{b}_{n\nearrow})) &\geq \frac{m_1}{m_n - 1} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\searrow})) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\nearrow})) \right] \\
&> \frac{m_1 - 1}{m_n} \left[\frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\searrow})) - \frac{d}{db} \ln F_1(\varphi_1(\bar{b}_{n\nearrow})) \right],
\end{aligned}
$$

which contradicts (14). Thus, the bracketed term must be zero, and it then follows immediately from (14) that the left-derivative of φ_n at $\bar{b}_n$, which we for simplicity denote $\varphi'_n(\bar{b}_n)$, must be zero as well (incidentally, it also follows that φ_1 does not have a kink at $\bar{b}_n$). ∎

Proof of Corollary 2. To prove the corollary it is sufficient to establish that $\bar{b}_1 > \bar{b}_1^s$. Let $EU_1^s(v)$ denote bidder 1's expected utility, as a function of his type, in

a symmetric auction against $m_1 - 1 \geq 1$ identical rivals. Myerson (1981) has shown that

$$EU_1^s(v) = EU_1^s(\underline{v}_1) + \int_{\underline{v}_1}^{v} q_1^s(x)dx,$$

where $q_1^s(x) = F_1(x)^{m_1-1}$ is the probability of winning for a bidder with type x. Of course, it must also hold that $EU_1^s(v) = (v - b_1^s(v))q_1^s(v)$, where $b_1^s(v)$ is the bidding strategy in the symmetric auction. Since $m_1 \geq 2$, it holds that $EU_1^s(\underline{v}_1) = 0$, since a type $\underline{v}_1$ bidder wins with probability zero. Hence, since $q_1^s(\overline{v}_1) = 1$,

$$\overline{v}_1 - \overline{b}_1^s = \int_{\underline{v}_1}^{\overline{v}_1} F_1(x)^{m_1-1}dx.$$

In the asymmetric auction we have

$$\overline{v}_1 - \overline{b}_1 = \int_{\underline{v}_1}^{\overline{v}_1} F_n\left(\varphi_n(b_1(x))\right)^{m_n} F_1(x)^{m_1-1}dx < \int_{\underline{v}_1}^{\overline{v}_1} F_1(x)^{m_1-1}dx = \overline{v}_1 - \overline{b}_1^s,$$

where $b_1(x)$ is bidder 1's equilibrium bidding strategy in the asymmetric auction. It follows that $\overline{b}_1 > \overline{b}_1^s$. ∎

Proof of Proposition 2. The relationship in Proposition 1 characterizes a necessary condition on any candidate $(\widehat{v}, \overline{b}_n)$ pair. The next step is to use mechanism design arguments to derive a second necessary condition. The final step combines these two conditions to establish Proposition 2.

As in any mechanism design argument, the equilibrium allocation plays an important role. Thus, let $q_i(v)$ denote the probability that a bidder in group i, $i = 1, n$, wins the auction if his type is v. Letting $EU_i(v)$ denote such a bidder's expected utility, Myerson (1981) has shown that

$$EU_i(v) = EU_i(\underline{v}_i) + \int_{\underline{v}_i}^{v} q_i(x)dx.$$

In the setting in Proposition 2, it is easily seen that $EU_i(\underline{v}_i) = 0$ (recall that $\underline{v}_i = 0$).

Consider now the highest types, $\overline{v}_1$ and $\overline{v}_n$, respectively. First, observe that

$$
\begin{aligned}
EU_1(\overline{v}_1) &= EU_1(\widehat{v}) + \int_{\widehat{v}}^{\overline{v}_1} q_i(x) dx \\
&= \left(\widehat{v} - \overline{b}_n\right) \left(\frac{\widehat{v}}{\overline{v}_1}\right)^{m_1-1} + \int_{\widehat{v}}^{\overline{v}_1} \left(\frac{x}{\overline{v}_1}\right)^{m_1-1} dx,
\end{aligned}
$$

since type $x \geq \widehat{v}$ outbids all group n bidders with probability one and thus wins if all rival bidders in group i have types that are below x. Conveniently, this expression does not require any knowledge of $q_1(x)$ for $x < \widehat{v}$. Integrating now yields the conclusion that

$$
\int_{\underline{v}_1}^{\overline{v}_1} q_1(x) dx = \left(\widehat{v} - \overline{b}_n\right) \left(\frac{\widehat{v}}{\overline{v}_1}\right)^{m_1-1} + \frac{1}{m_1} \frac{\overline{v}_1^{m_1} - \widehat{v}^{m_1}}{\overline{v}_1^{m_1}}. \tag{15}
$$

Similarly, since

$$
EU_n(\overline{v}_n) = \left(\overline{v}_n - \overline{b}_n\right) \left(\frac{\widehat{v}}{\overline{v}_1}\right)^{m_1},
$$

it follows that

$$
\int_{\underline{v}_n}^{\overline{v}_n} q_1(x) dx = \left(\overline{v}_n - \overline{b}_n\right) \left(\frac{\widehat{v}}{\overline{v}_1}\right)^{m_1}. \tag{16}
$$

The ex ante probability that any given bidder wins the auction takes a particularly useful form when distributions are uniform, since

$$
\int_{\underline{v}_i}^{\overline{v}_i} q_i(x) f_i(x) dx = \frac{1}{\overline{v}_i} \int_{\underline{v}_i}^{\overline{v}_i} q_i(x) dx.
$$

Since the auction has no reserve price, the item will be sold for sure. In other words, the ex ante winning probabilities must aggregate to one, or

$$
m_1 \frac{1}{\overline{v}_1} \int_{\underline{v}_1}^{\overline{v}_1} q_1(x) dx + m_n \frac{1}{\overline{b}_n} \int_{\underline{v}_n}^{\overline{v}_n} q_n(x) dx = 1. \tag{17}
$$

Combining (15) and (16) with (17) yields the necessary condition that

$$
\overline{b}_n = \frac{m}{\overline{b}_n m_1 + m_n \widehat{v}} \overline{v}_n \widehat{v} \tag{18}
$$

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for $\widehat{v} \in [0, \overline{v}_1]$, or, stated differently,

$$\widehat{v} = \frac{\overline{v}_n m_1 \overline{b}_n}{m_n \left(\overline{v}_n - \overline{b}_n\right) + \overline{v}_n \left(m_1 - 1\right)} \tag{19}$$

with the restriction that $\overline{b}_n$ is such that $\widehat{v} \in [0, \overline{v}_1]$.

In summary, any equilibrium $(\widehat{v}, \overline{b}_n)$ pair must satisfy both (19) and (8). Thus, the next step is to characterize what turns out to be the unique $(\widehat{v}, \overline{b}_n)$ pair that satisfies both conditions. First, note that the right hand side of (19) is strictly increasing in $\overline{b}_n$ and ranges from 0 to $\frac{m_1}{m_1-1}\overline{v}_n$ as $\overline{b}_n$ increases from 0 to $\overline{v}_n$. However, the term $\frac{m_1}{m_1-1}\overline{v}_n - \frac{1}{m_1-1}\overline{b}_n$ on the right hand side of (8) is strictly decreasing in $\overline{b}_n$ and ranges from $\frac{m_1}{m_1-1}\overline{v}_n$ to $\overline{v}_n$ as $\overline{b}_n$ increases from 0 to $\overline{v}_n$. Thus, the two equations (i.e. (19) and $\widehat{v} = \frac{m_1}{m_1-1}\overline{v}_n - \frac{1}{m_1-1}\overline{b}_n$) must have a unique intersection with $\overline{b}_n \in (0, \overline{v}_n)$. We first identify this intersection and then subsequently check whether it satisfies the feasibility condition that $\widehat{v} \in [0, \overline{v}_1]$. Equalizing these two equations yields a quadratic equation in $\overline{b}_n$. The larger root can be ruled out, since it yields the conclusion that $\overline{b}_n > \overline{v}_n$. The smaller root is $\overline{b}_n = \kappa(m_1, m_n)\overline{v}_n$, for which $\widehat{v} = \tau(m_1, m_n)\overline{v}_n$. This candidate satisfies the final feasibility condition that $\widehat{v} \leq \overline{v}_1$ if and only if $\tau(m_1, m_n) \leq \frac{\overline{v}_1}{\overline{v}_n}$. This proves the second part of the proposition. If $\tau(m_1, m_n) > \frac{\overline{v}_1}{\overline{v}_n}$, the condition that $\widehat{v} \leq \overline{v}_1$ instead binds. Nevertheless, (19) and (18) must be satisfied. The latter establishes the characterization in the first part of the proposition.

Continuity follows from the continuity of (19) and (8). Of course this implies that when $\tau(m_1, m_n)$ is identical to $\frac{\overline{v}_1}{\overline{v}_n}$, the equilibrium pair $(\widehat{v}, \overline{b}_n)$ in the two parts of the proposition coincide. ∎

Proof of Corollary 3. If it is initially the case that $\tau(m_1, m_n) > \frac{\overline{v}_1}{\overline{v}_n}$, then the inequality remains true as $\overline{v}_n$ increases. Thus, the first part of Proposition 2 applies. In particular, $\widehat{v} = \overline{v}_1$ remains unchanged, while $\overline{b}_n$ strictly increases. Assume next that $\tau(m_1, m_n) \leq \frac{\overline{v}_1}{\overline{v}_n}$ initially. There are two cases. If the inequality remains true as $\overline{v}_n$ increases, then it follows directly from the characterization in the second part of the Proposition 2 that both $\widehat{v}$ and $\overline{b}_n$ strictly increases. If the inequality is reversed, then $\widehat{v}$ must increase from some value (weakly) below $\overline{v}_1$ to precisely $\overline{v}_1$. It follows from (18) that $\overline{b}_n$ strictly increases. This proves the first part of the corollary.

For the second part, note that (8) is independent of m_n, while the curve described in (19) shifts down when m_n increases. Hence, the intersection of the two curves

moves south-east in $(\bar{b}_n, \widehat{v})$ space. This proves the second part of the corollary. ∎

Proof of Corollary 4.   The first part follows immediately from the equilibrium characterization in Proposition 2. For the second part, note that if If $\frac{\bar{v}_1}{\bar{v}_n} \leq \tau(m_1, m_n)$ then $\widehat{v}$ remains unchanged at $\bar{v}_1$ as m_1 increases. However, $\bar{b}_n$ strictly increases. If $\frac{\bar{v}_1}{\bar{v}_n} > \tau(m_1, m_n)$ then $\bar{b}_n = \kappa(m_1, m_n)\bar{v}_n$, $\widehat{v} = \tau(m_1, m_n)\bar{v}_n$. The proof concludes by recalling that $\kappa(m_1, m_n)$ is strictly increasing in m_1 and that $\tau(m_1, m_n)$ is strictly decreasing in m_1. ∎

Proof of Proposition 3.   Let bidder i's inverse bidding strategy be denoted $\varphi_i^F(b)$ and $\varphi_i^G(b)$ in the two scenarios where distributions are (F_1, F_n) and (G_1, G_n), respectively (recall the assumption that equilibrium is unique). The case where $G_1 = F_1$ and $G_n = F_n$ is uninteresting. Thus, assume in the remainder that $G_1 \succ F_1$ and/or $G_n \succ F_n$. The system in (2) can be written as

$$\varphi_i'(b) = \frac{1}{n-1} \frac{F_i(\varphi_i(b))}{f_i(\varphi_i(b))} \left[\frac{m_j}{\varphi_j(b) - b} - \frac{m_j - 1}{\varphi_i(b) - b} \right] \tag{20}$$

where $j \neq i$, $i = 1, 2$.

Assume by contradiction that $\bar{b}_n^G < \bar{b}_n^F$, which by Proposition 1 implies $\widehat{v}^G \geq \widehat{v}^F$. Hence, $\varphi_i^G(\bar{b}_n^G) > \varphi_i^F(\bar{b}_n^G)$, $i = 1, 2$. Now move leftwards (reducing b) from $\bar{b}_n^G$. Let $b' > r$ denote the first (i.e. highest) bid for which $\varphi_i^G(b') = \varphi_i^F(b')$ for some (or both) i, if it exists. If it exists, there are two possibilities. One possibility is that the crossing occurs at the same place for both $i = 1$ and $i = 2$, i.e. $\varphi_1^G(b') = \varphi_1^F(b')$ and $\varphi_n^G(b') = \varphi_n^F(b')$. The bracketed term in (20) is then the same for both scenarios. However, since $G_i \succ F_i$ for some i, it follows that $\varphi_i^{G\prime}(b') < \varphi_i^{F\prime}(b')$. However, this contradicts the fact that $\varphi_i^G(b) > \varphi_i^F(b)$ to the right of b'. The other possibility is that $\varphi_i^G(b') = \varphi_i^F(b')$ but $\varphi_j^G(b') > \varphi_j^F(b')$, $j \neq i$. The conclusion is again that $\varphi_i^{G\prime}(b') < \varphi_i^{F\prime}(b')$, because the bracketed term in (20) is smaller (and the first term is no larger) when distributions are (G_1, G_n) compared to (F_1, F_n). The same contradiction is thus achieved. It now follows that $\varphi_i^G(b) > \varphi_i^F(b)$ for all $b \in (r, \bar{b}_n^G]$, $i = 1, 2$. Assuming that $G_1 \succ F_1$ (the proof is similar if $G_n \succ F_n$ instead) it follows from (1)

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that

$$\frac{d}{db} \ln F_n(\varphi_n^F(b))^{m_n} F_1(\varphi_1^F(b))^{m_1-1} = \frac{1}{\varphi_1^F(b) - b}$$

$$> \frac{1}{\varphi_1^G(b) - b}$$

$$= \frac{d}{db} \ln G_n(\varphi_n^G(b))^{m_n} G_1(\varphi_1^G(b))^{m_1-1},$$

or

$$\frac{d}{db} \left[\ln \left(\frac{F_n(\varphi_n^F(b))}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{F_1(\varphi_1^F(b))}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} \right] > \frac{d}{db} \left[\ln \left(\frac{G_n(\varphi_n^G(b))}{G_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(b))}{G_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1} \right]$$

for all $b \in (r, \bar{b}_n^G]$. Evaluated at $b = \bar{b}_n^G$, the bracketed term on either side of the inequality are both zero. Since $r > \underline{v}$, the bracketed term on the left converges to a finite value as $b \to r$. Moreover, since the bracketed term on the left is steeper in b than the bracketed term on the right, the latter must also converge to a finite value, with

$$\ln \left(\frac{F_n(\varphi_n^F(r))}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{F_1(\varphi_1^F(r))}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} < \ln \left(\frac{G_n(\varphi_n^G(r))}{G_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(r))}{G_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1}$$

Since $(\varphi_1^F, \varphi_n^F)$ are equilibrium strategies, it must hold that $\varphi_i^F(r) = r$, $i = 1, 2$. If $(\varphi_1^G, \varphi_n^G)$ are equilibrium strategies as well, then it also holds that $\varphi_i^G(r) = r$, $i = 1, 2$, and we have

$$\left(\frac{F_n(r)}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{F_1(r)}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} < \left(\frac{G_n(r)}{G_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(r)}{G_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1}$$

or

$$\left(\frac{G_n(\varphi_n^G(\bar{b}_n^G))}{F_n(\varphi_n^F(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(\bar{b}_n^G))}{F_1(\varphi_1^F(\bar{b}_n^G))} \right)^{m_1-1} < \left(\frac{G_n(r)}{F_n(r)} \right)^{m_n} \left(\frac{G_1(r)}{F_1(r)} \right)^{m_1-1}$$

which, since $\varphi_i^G(\bar{b}_n^G) > \varphi_i^F(\bar{b}_n^G)$, implies

$$\left(\frac{G_n(\varphi_n^G(\bar{b}_n^G))}{F_n(\varphi_n^G(\bar{b}_n^G))} \right)^{m_n} \left(\frac{G_1(\varphi_1^G(\bar{b}_n^G))}{F_1(\varphi_1^G(\bar{b}_n^G))} \right)^{m_1-1} < \left(\frac{G_n(r)}{F_n(r)} \right)^{m_n} \left(\frac{G_1(r)}{F_1(r)} \right)^{m_1-1}$$

However, the assumption that $G_i \succ F_i$ ($G_i = F_i$) is equivalent to $\frac{d}{dv} \frac{G(v)}{F(v)} > 0$ ($\frac{d}{dv} \frac{G(v)}{F(v)} = 0$). Consequently, the above inequality must be violated. In other words, $(\varphi_1^G, \varphi_n^G)$ cannot form an equilibrium. ∎